From last time

- More on electric potential and connection to E-field
- How to calculate E-field from V
- Capacitors and Capacitance

Today:

- More on Capacitors and Capacitance
- Energy stored in Capacitors
- Current
- Resistance
- Ohm’s Law
How do we charge a capacitor?

✦ Battery has fixed electric potential difference across its terminals
✦ Conducting plates connected to battery terminals produce a charge separation on plates
✦ Electrons move from negative battery terminal to right plate and from left plate to positive battery terminal
✦ This charge motion requires work. The work is supplied by the battery.
✦ Between plates there is an E-field and a potential difference is established

\[ \Delta V_{\text{plates}} = \Delta V_{\text{battery}} \]

Infinite plates: we normally mean to neglect border effects (uniform field)
Capacitance

\[ E_{\text{left}} + E_{\text{right}} = \frac{\sigma}{2\varepsilon_o} + \frac{\sigma}{2\varepsilon_o} = \frac{\sigma}{\varepsilon_o} \]

\[ \Delta V = \frac{1}{C} Q \]

Work of the electric force to move q from positive to negative plate =

\[ W_{el} = -\Delta U = -q(V_+ - V_-) = q \int_{0}^{d} \vec{E} \cdot d\vec{s} = qEd \Rightarrow \Delta V = Ed \]

\[ \Delta V = \frac{\sigma}{\varepsilon_0} d = \frac{Qd}{\varepsilon_0 A} = \frac{Q}{C} \]

\[ C = \frac{\varepsilon_o A}{d} \]

This is a geometrical factor
Spherical capacitor

Charge $Q$ moved from outer to inner sphere

Gauss’ law says $E = kQ/r^2$

between the inner sphere and the shell

Potential difference

Along path shown

$$\Delta V = \int_{a}^{b} \frac{kQ}{r^2} = -kQ \left[ \frac{1}{r} \right]_{a}^{b} = kQ \left( \frac{1}{a} - \frac{1}{b} \right)$$

$$C = \frac{Q}{\Delta V} = \frac{1}{k \left( \frac{1}{a} - \frac{1}{b} \right)}$$

Gaussian surface to find $E$

Path to find $\Delta V$
**Energy stored in capacitors**

- During the charging of a capacitor, when a charge $q$ is on the plates, the work needed to transfer further $dq$ from one plate to the other is:

$$dW = \Delta V dq = \frac{q}{C} dq$$

- The total work required to charge the capacitor is

$$W = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C}$$

- The energy stored in any capacitor is:

For a parallel capacitor:

$$U = \frac{Q^2}{2C} = \frac{1}{2} Q\Delta V = \frac{1}{2} C(\Delta V)^2 \quad U = 1/2 \ \varepsilon_o A dE^2$$
Energy density

- The energy stored per unit volume is
  \[ \frac{U}{(Ad)} = \frac{1}{2} \varepsilon_o E^2 \]
- This is a fundamental relationship for the local energy stored in an electric field
- Not restricted to capacitors: E-field can be from any source
- Interpretation: energy is stored in the field
- Capacitors are devices to store electric energy and charge (used in radio receivers, filters of power supplies, electronic flashes)
A parallel plate capacitor given a charge $q$. The plates are then pulled a small distance further apart. Which of the following apply to the situation after the plates have been moved?

A) The charge decreases  
B) The capacitance increases  
C) The electric field increases  
D) The voltage between the plates increases  
E) The energy stored in the capacitor increases

The charge cannot decrease because C is isolated

$C = \varepsilon_0 \frac{A}{d} \Rightarrow C$ decreases!

$E = \frac{Q}{(\varepsilon_0 A)} \Rightarrow E$ constant

$V = Ed \Rightarrow V$ increases

$U = \frac{Q^2}{2C}$  
$U$ increases
Human capacitors: cell membranes

Lipid bilayers form a capacitor with charge separation at about 7-8nm
Modeling a cell membrane

- Charges are +/- ions instead of electrons (atoms with different numbers of electrons and protons)
- Charge motion is through cell membrane (ion channels) rather than through wire
- Channels are selective for each kind of ion
- Across the cell membrane about $\Delta V=-70$ mV (resting potential) since more Na+ outside and gates are closed while K+ ions diffuse out and in at the same rate
- There is more positive charge outside the cell

Capacitance:

$$\frac{\varepsilon_o A}{d} = \frac{\left(8.85 \times 10^{-12} \text{ } F/m\right) 4\pi \left(50 \times 10^{-6} \text{ } m\right)^2}{8 \times 10^{-9} \text{ } m} = 3.5 \times 10^{-11} \text{ } F = 35 \text{ } pF$$

100 µm sphere surface area $\sim 3 \times 10^{-4}$ cm$^2$

$K^+ \quad Na^+ \quad Na^+$
Extracellular fluid

$+$ $+$ $+$ $+$ $+$ $+$

$Na^+ \quad K^+$
Cytoplasm

$-$ $-$ $-$ $-$ $-$ $-$

$7-8$ nm

$\Delta V \sim -0.07$ V
Depolarization in a nerve cell
An influx of Na\textsuperscript{+} ions thru Na\textsuperscript{+} channel can cause a depolarization of the membrane. The potential difference becomes about 0.03 mV. Potential change = about 0.1 V

About 100 channels/µm\textsuperscript{2} (each has radius of 50 µm)

How many ions flow per channel?

Charge xfer required \( \Delta Q = C \Delta V = (35 \text{ pF})(0.1\text{V}) = (35 \times 10^{-12} \text{C/V})(0.1\text{V}) = 3.5 \times 10^{-12} \text{Coulombs} \)

\( 3.5 \times 10^{-12} \text{C} / (1.6 \times 10^{-19} \text{C/ion}) = 2.2 \times 10^7 \text{ ions flow} \)

\( (100 \text{ channels/µm}^2) \times 4\pi (50 \text{ µm})^2 = 3.14 \times 10^6 \text{ ion channels} \)

\( \text{Ion flow / channel} = \frac{(2.2 \times 10^7 \text{ ions})}{3.14 \times 10^6 \text{ channels}} \approx 7 \text{ ions/channel} \)
Charge motion: current

• Cell membrane capacitor:
  Found ~7 ions flow through ion each ion channel to depolarize membrane
• Occurs in ~ 1 ms = 0.001 sec.
• This is an electric current \( I \): amount of charge per unit time flowing through a plane perpendicular to charge motion
• Units: 1 Coulomb / sec = 1 Ampere

  - Average current:
    \[ I_{av} = \frac{\Delta Q}{\Delta t} \]
  - Instantaneous value:
    \[ I = \frac{dQ}{dt} \]

Conventionally we assume:
  current in direction of + charge particles though electrons in reality flow in a wire
Current in a conductor wire

Battery produces E-field in wire

- Current flows in response to E-field

Average current:

\[ I_{av} = \frac{\Delta Q}{\Delta t} \]

- n = number of electrons/volume
- Electrons travel distance \( L = v_d \Delta t \) in time \( \Delta t \)
- # of electrons: \( n \times \text{volume} = nAL = nA v_d \Delta t \)
- \( I_{av} = \frac{\Delta Q}{\Delta t} = nqAL v_d /L \)

Current density

- \( J = I/A \) (A/m²)
- \( J = nq v_d \) (direction of + charge carriers)

\( v_d = 10^{-4} \text{ m/s} \)
Ohm’s law

- When an electric field is applied free electrons experience an acceleration (opposite to $E$): $a = F/m = qE/m$
- Calling $t$ the mean time between 2 collisions electron-atoms electrons achieve a drift velocity:

$$v_d = at = \frac{qE}{m} \Rightarrow \vec{j} = nq\vec{v}_d = \left(\frac{nq^2t}{m}\right)\vec{E}$$

- **Ohm’s Law:** for many materials, the current density is proportional to the electric field producing the current $J = \sigma E$

- $\sigma = \text{conductivity}$ of the conductor
Ohm’s Law

- Ohm’s Law: \( J = \sigma E \) or \( E = \rho J \)
- \( \rho = \text{resistivity} \)

\[
E = \frac{\Delta V}{L} = \rho \frac{I}{A}
\]

- Define: \( R = \rho L/A \)
- **Resistance** in ohms (\( \Omega \))
- Ohm’s law becomes: \( \Delta V = R I \)

- ohmic device: relationship current and voltage is linear
Quick Quiz

Two cylindrical conductors are made from the same material. They are of equal length but one has twice the diameter of the other.

A. \( R_1 < R_2 \)
B. \( R_1 = R_2 \)
C. \( R_1 > R_2 \)

\[ R = \rho \frac{L}{A} \]

Current flow ~ uniform. More cross-sectional area means more current flowing -less resistance.
Dependence on Temperature

\[ R = \rho \frac{L}{A} \]

- \( \rho \) of a conductor varies approximately linearly with \( T \)
  - \( \rho_o \) is the resistivity at \( T_o = 20^\circ \text{C} \)
  - \( \alpha \) = temperature coefficient in SI units of \( ^\circ \text{C}^{-1} \)

\[ \rho = \rho_o [1 + \alpha (T - T_o)] \]

- The higher \( T \) the greater atomic vibrations that increases collision probability

Similarly: \( R = R_o [1 + \alpha (T - T_o)] \)
But other materials…

- **Semiconductors:**
  - \( \alpha \) is negative \( \Rightarrow \rho \) decreases for increasing \( T \)
  - \[ \rho = \rho_0 (1 + \alpha \Delta T) \]

- **Superconductors**
  - Below a certain temperature, \( T_c = \text{critical temperature} \) \( \rho \) is zero
  - Once a current is set up in a superconductor, it persists without any applied voltage since \( R = 0 \)
Resistivity

- **Resistivity:**
  - $\rho = 1 / \sigma$
  - SI units of $\Omega \cdot m$
- The resistance depends on resistivity and geometry:

$$R = \rho \frac{l}{A}$$

- Resistors control the current level in circuits
- Resistors can be composite or wire-wound

### Table 27.1

<table>
<thead>
<tr>
<th>Material</th>
<th>Resistivity$^a$(Ω · m)</th>
<th>Temperature Coefficient$^b$ $\alpha$[(°C)$^{-1}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silver</td>
<td>$1.59 \times 10^{-8}$</td>
<td>$3.8 \times 10^{-3}$</td>
</tr>
<tr>
<td>Copper</td>
<td>$1.7 \times 10^{-8}$</td>
<td>$3.9 \times 10^{-3}$</td>
</tr>
<tr>
<td>Gold</td>
<td>$2.44 \times 10^{-8}$</td>
<td>$3.4 \times 10^{-3}$</td>
</tr>
<tr>
<td>Aluminum</td>
<td>$2.82 \times 10^{-8}$</td>
<td>$3.9 \times 10^{-3}$</td>
</tr>
<tr>
<td>Tungsten</td>
<td>$5.6 \times 10^{-8}$</td>
<td>$4.5 \times 10^{-3}$</td>
</tr>
<tr>
<td>Iron</td>
<td>$10 \times 10^{-8}$</td>
<td>$5.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>Platinum</td>
<td>$11 \times 10^{-8}$</td>
<td>$3.92 \times 10^{-3}$</td>
</tr>
<tr>
<td>Lead</td>
<td>$22 \times 10^{-8}$</td>
<td>$3.9 \times 10^{-3}$</td>
</tr>
<tr>
<td>Nichrome$^c$</td>
<td>$1.50 \times 10^{-6}$</td>
<td>$0.4 \times 10^{-3}$</td>
</tr>
<tr>
<td>Carbon</td>
<td>$3.5 \times 10^{-5}$</td>
<td>$-0.5 \times 10^{-3}$</td>
</tr>
<tr>
<td>Germanium</td>
<td>$0.46$</td>
<td>$-48 \times 10^{-3}$</td>
</tr>
<tr>
<td>Silicon</td>
<td>$640$</td>
<td>$-75 \times 10^{-3}$</td>
</tr>
<tr>
<td>Glass</td>
<td>$10^{10}$ to $10^{14}$</td>
<td></td>
</tr>
<tr>
<td>Hard rubber</td>
<td>$\sim 10^{13}$</td>
<td></td>
</tr>
<tr>
<td>Sulfur</td>
<td>$10^{15}$</td>
<td></td>
</tr>
<tr>
<td>Quartz (fused)</td>
<td>$75 \times 10^{16}$</td>
<td></td>
</tr>
</tbody>
</table>

$^a$ All values at 20°C.
$^b$ See Section 27.4.
$^c$ A nickel–chromium alloy commonly used in heating elements.
Resistors in Series

- $I_1 = I_2 = I$
- $\Delta V = \Delta V_1 + \Delta V_2$
- $R_{eq} = R_1 + R_2$

Neglect resistance of wires in circuits respect to $R$ resistance

$$\Delta V = \Delta V_1 + \Delta V_2 = R_1 I_1 + R_2 I_2 = (R_1 + R_2)I = R_{eq} I$$
Resistors in parallel

- $\Delta V = \Delta V_1 = \Delta V_2$
- $I = I_1 + I_2$

\[
\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}
\]

\[
I = I_1 + I_2 = \frac{\Delta V_1}{R_1} + \frac{\Delta V_2}{R_2} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\Delta V = \frac{\Delta V}{R_{eq}}
\]

Add areas

$R = \rho \frac{L}{A}$
Current Conservation: $1^{st}$ Kirchoff’s law

- Junction Rule: $\Sigma I_{in} = \Sigma I_{out}$
  - A statement of Conservation of Charge
Electrical power

- As a charge $\Delta Q$ moves from $a$ to $b$, the electric potential energy of the system increases by $\Delta Q \Delta V$ and the chemical energy in the battery decreases by this same amount.

- As the charge moves through the resistor ($c$ to $d$), the system loses this electric potential energy during collisions of the electrons with the atoms of the resistor.

- This energy is transformed into internal energy in the resistor (vibrational motion of the atoms in the resistor).

- Power in units of Watts = J/s

$$P = \frac{\Delta U}{\Delta t} = \frac{\Delta Q}{\Delta t} \Delta V$$

$$P = I \Delta V = RI^2 = \frac{\Delta V^2}{R}$$
Quick Quiz

How does brightness of bulb B compare to that of A?

A. B brighter than A

B. B dimmer than A

C. Both the same

In A: \[ I_A = \frac{\varepsilon}{R} \Rightarrow P_A = R I_A^2 = \frac{\varepsilon^2}{R} \]

In B: \[ I_B = \frac{\varepsilon}{2R} \Rightarrow P_B = \frac{\varepsilon^2}{4R} \]
Dielectrics in capacitors

- Membrane is not really empty
- It has molecules inside that respond to electric field.
- The molecules in the membrane can be polarized

$E = \frac{E_0}{\kappa}$ and $V = \frac{V_0}{\kappa}$

$\kappa > 1$

The capacitance increases $C = \kappa C_0$

In cells lipid belayer as $\kappa = 10$ so 70 ions flow/channel