HW1 Possible Solutions

Notice numbers may change randomly in your assignments and you may have to recalculate solutions for your specific case.

Tipler 14.P.003
An object attached to a spring has simple harmonic motion with an amplitude of \( A = 3.00 \) cm. When the object is \( x = 1.73 \) cm from the equilibrium position, what fraction of its total energy is potential energy?

Solution:
Total energy: \( E = \frac{1}{2} kA^2 \), with \( k \) = elastic constant

Potential energy: \( U(x) = \frac{1}{2} kx^2 \)

\[
\frac{U(x)}{E} = \frac{\frac{1}{2} kx^2}{\frac{1}{2} kA^2} = \frac{x^2}{A^2} = 0.33 \text{cm} = \text{1 third}
\]

Tipler 14.P.004
An object attached to a spring exhibits simple harmonic motion with an amplitude of \( A=9.6 \) cm. How far from equilibrium will the object be when the system’s potential energy is equal to its kinetic energy?

Solution:

The potential energy and kinetic energy are equal when

\[
E_k = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \delta) = U = \frac{1}{2} kA^2 \cos^2(\omega t + \delta)
\]

Sin and cos functions are equal for an angle of 45 degrees or \( \pi/4 \).

Hence: \( x(t) = A\cos\frac{\pi}{4} = 6.8 \text{cm} \) while the position of equilibrium is the position where \( U(x) = 0 \) and so where \( x(t)=0 \). So the object will be 6.8 cm far from equilibrium.
The position of a particle is given by \( x = (7.0 \text{ cm}) \cos 6\pi t \), where \( t \) is in seconds.

(a) What is the frequency?
(b) What is the period?
(c) What is the amplitude of the particle's motion?
(d) What is the first time after \( t = 0 \) that the particle is at its equilibrium position?

In what direction is it moving at that time?
1) in the negative direction
2) in the positive direction

(a) The position of the particle is given by:
\[
x(t) = Acos(\omega t) = 7.0 \text{ cm} \cos 6\pi t \Rightarrow \omega = 6\pi = 2\pi f \Rightarrow f = 3.0 \text{ Hz}
\]

(b) 
\[
T = \frac{2\pi}{\omega} = \frac{1}{f} = 0.3 \text{ s}
\]

(c) Comparing the formula given in the problem to the one in (a): \( A = 7 \text{ cm} \)

(d) The equilibrium position happens when \( U \) is minimum: \( U = 0 \)

when 
\[
U(x) = \frac{1}{2} kx^2 \Rightarrow \frac{dU}{dx} = 0 \Rightarrow x = 0 \Rightarrow \cos(6\pi t) = 0 \Rightarrow 6\pi t = (2n + 1)\frac{\pi}{2}
\]

\[n = 0 \Rightarrow t = \frac{1}{12}s = 0.083s\]

1) In the negative direction: in fact differentiate \( x \) to find the speed and calculate it for \( t = 1/12 \) s

\[
v = \frac{d}{dt}[A \cos(\omega t)] = -A \omega \sin(\omega t) \Rightarrow v(1/12s) = -(42 \text{ cm/s}) \sin(6\pi/12) = -42 \text{ cm/s} < 0
\]

Since the speed is negative, the particle moves in the direction \(-x\) at \( t = 1/12 \) s.

**Tipler 14.P.039**

A particle moves at a constant speed of 150 \text{ cm/s} in a circle of radius \( R=75 \text{ cm} \) centered at the origin in an anticlockwise direction.

(a) Find the frequency and period of the \( x \) component of its
position.
(b) Write an expression for the \( x \) component particle's position as a function of time \( t \), assuming that the particle is located on the \(+y\) axis at time \( t = 0 \). (Keep your constants in units of cm and s. Use \( \pi \) for \( \pi \) and \( t \) as necessary.)

**Solution:**

(a) \( v = \omega R \Rightarrow f = \frac{\omega}{2\pi} = \frac{v}{2\pi R} = 0.32 \text{Hz} \) (the frequency is the reciprocal of the period = the time the particle takes to make a full round)

\[
T = \frac{1}{f} = 3.1 \text{s}
\]

(b) \( x = A\cos(\omega t + \delta) \)

\( \delta = \pi/2 \) since at \( t=0 \) the particle must be on the \(+y\) axis that is at \( x(t=0) = 0 \), \( \omega = \frac{2\pi}{T} = 2.0 \text{rad/s} \) and \( A = R = 75 \text{ cm} \)

\[
\Rightarrow x(t) = 75\cos(2t + \pi/2) \text{ in cm and } t \text{ in s.}
\]

Since the problem says the particle goes in an anticlockwise direction, we can check if the expression we chose for \( x \) is correct by calculating the velocity that should be negative at time \( t = 0 \) then the particle is along \(+y\).

\[
v_y = \frac{dx}{dt} = -A\omega\sin(\omega t + \pi/2)
\]

\( t = 0 \Rightarrow v_y = -A\omega \)

Since the velocity is negative we have initially chosen the right sign for \( x \). So the answer is

\[
x = A\cos\left(\omega t + \frac{\pi}{2}\right) = -A\sin\omega t
\]

**Tipler 14.P.043**

A 1.5 kg object oscillates with simple harmonic motion on a spring of force constant \( k = 585 \text{ N/m} \). Its maximum speed is 80 cm/s.

(a) What is the total mechanical energy?
(b) What is the amplitude of the oscillation?

**Solution:**
(a) The speed is maximum at the equilibrium position where the potential energy is minimum: 
\[ E = \frac{1}{2} kA^2 = E_{k,\text{max}} = \frac{1}{2} mv_{\text{max}}^2 = 0.48J \]

(b) \[ A = \sqrt{\frac{mv_{\text{max}}^2}{k}} = \sqrt{\frac{2E_{\text{tot}}}{k}} = 4.05cm \]

**Tipler 14.P.046**
A 3.0 kg object oscillates on a spring with an amplitude of 7.5 cm. Its maximum acceleration is 3.50 m/s². Find the total energy.

**Solution:**
According to Newton’s 2nd law: 
\[-kx = ma, \text{ hence} \]
\[ a = -\omega^2 x \Rightarrow a_{\text{max}} = -\omega^2 x_{\text{min}} = \omega^2 A \]
\[ E = \frac{1}{2} kA^2 = \frac{1}{2} m\omega^2 A^2 = \frac{1}{2} ma_{\text{max}} A = 0.394J \]

**Tipler 14.P.047**
A 2.5 kg object is attached to a horizontal spring of force constant \( k = 4.5 \text{ kN/m} \). The spring is stretched 10 cm from equilibrium and released.
(a) Find the frequency of the motion.
(b) Find the period.
(c) Find the amplitude.
(d) Find the maximum speed.
(e) Find the maximum acceleration.
(f) When does the object first reach its equilibrium position? What is its acceleration at this time?

**Solution:**

(a) \[ f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = 6.75Hz \]
(b) $T = \frac{1}{f} = 0.15\, s$

(c) $A = 10\, \text{cm}$

(d) $E = \frac{1}{2} kA^2 = E_{k,\text{max}} = \frac{1}{2} m v_{\text{max}}^2 \Rightarrow v_{\text{max}} = \sqrt{\frac{k}{m}} A = \omega A = 2\pi f A = 4.24\, m/s$

(e) $a_{\text{max}} = A \omega^2 = v_{\text{max}}^2 \omega = 2\pi f v_{\text{max}} = 179.82\, m/s^2$

(f) When the potential energy becomes minimum, so when $x = 0$

$U = \frac{1}{2} kx^2 = 0 \Rightarrow x = A \cos(\omega t) = 0 \Rightarrow \omega t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2(2\pi f)} = \frac{1}{4f} = 37.04\, ms$

$a = -\omega^2 x = 0$

**Tipler 14.P.061**
The period of a simple pendulum is $3.0\, s$ at a point where $g = 9.81\, m/s^2$. What would be the period of this pendulum if it were on the Moon, where the acceleration due to gravity is one-sixth that on Earth?

**Solution:**

$$T = 2\pi \sqrt{\frac{L}{g}} \Rightarrow \frac{T}{T_{\text{Moon}}} = \sqrt{\frac{g_{\text{Moon}}}{g}} = \sqrt{\frac{1}{6}} \Rightarrow T_{\text{Moon}} = \sqrt{6} T = 7.35\, s$$

**Tipler 14.P.076**
A $2\, \text{kg}$ object oscillates with an initial amplitude of $3\, \text{cm}$ on a spring of force constant $k = 405\, \text{N/m}$.

(a) Find the period.

(b) Find the total initial energy.

(c) If the energy decreases by $1\%$ per period, find the damping constant $b$ and the $Q$ factor.

**Solution:**

(a) $T = 2\pi \sqrt{\frac{m}{k}} = 0.442\, s$

(b) $E = \frac{1}{2} kA^2 = 0.182\, J$
(c) The damping constant $b$ is connected to $Q$ by the relation:

$$Q = \omega_0 \tau \quad \text{where} \quad \tau = \frac{m}{b} \quad \text{is the time constant.}$$

$$\frac{|\Delta E|}{E} = 0.01 = \frac{T}{\tau} = \frac{Tb}{m} \Rightarrow b = \frac{0.01 \times m}{T} = 0.0045 \text{kg/s}$$

$$\frac{|\Delta E|}{E} = 0.01 = \frac{2\pi}{Q} \Rightarrow Q = 628$$

$$b = \frac{\omega_0 m}{Q} = \frac{2\pi m}{TQ} = \frac{2\pi m}{2\pi \sqrt{\frac{m}{k} Q}} = \sqrt{\frac{m}{k}}$$

$$b = 0.045 \text{ kg/s}$$

**Tipler 14.P.078**

An oscillator has a period of 2.1 s. Its amplitude decreases by 4% during each cycle.

(a) By how much does its energy decrease during each cycle?

(b) What is the time constant $\tau$?

(c) What is the $Q$ factor?

**Solution:**

(a) $E \propto A^2$

so if the amplitude decreases during each cycle by 4% the energy decreases by 8%. We have to differentiate the relationship between total energy and amplitude:

$$E \propto A^2 \Rightarrow \frac{|\Delta E|}{E} = \frac{|2\Delta A|}{A^2} = \frac{2|\Delta A|}{A} = 8\%$$

(b) $\frac{|\Delta E|}{E} = 0.08 = \frac{T}{\tau} \Rightarrow \tau = \frac{T}{0.08} = 26.25 \text{s}$

(c) $\frac{|\Delta E|}{E} = 0.08 = \frac{2\pi}{Q} \Rightarrow Q = 78.54$

**Tipler 14.P.086**

A damped oscillator loses 3.5% of its energy during each cycle.

(a) How many cycles elapse before half of its original energy is
dissipated? (round your answer to the closer integer)
(b) What is its \( Q \) factor?
(c) If the natural frequency is 100 Hz, what is the width of the resonance curve when the oscillator is driven?

**Solution:**

(a) 
\[
\frac{\Delta E}{E} = 0.035 = \frac{2\pi}{Q} \Rightarrow Q = \frac{2\pi}{0.035} 
\]
\[
\tau = \frac{1}{f} \Rightarrow t = 0.035 \times f 
\]
\[
E = E_0 e^{-t/\tau} \Rightarrow E_0 = \frac{E}{e^{-t/\tau}} 
\]
\[
t = \ln(2) \times \tau = \ln(2) \times \frac{1}{0.035 \times f} \Rightarrow N = \frac{t}{T} = tf = \ln(2) \times \frac{f}{0.035 \times f} = 19.8 
\]

where \( N \) is the number of cycles that occur before the energy is halved. \( N \) is about 20 cycles.

(b) 
\[
\frac{\Delta E}{E} = 0.035 = \frac{2\pi}{Q} \Rightarrow Q = 179.5 
\]

(c) 
\[
\frac{\Delta \omega}{\omega_0} = \frac{1}{Q} \Rightarrow \Delta \omega = \frac{\omega_0}{Q} = \frac{2\pi f}{Q} = 3.5 \text{ rad/s} 
\]