Shooting a Block up an Incline

**Description:** Shoot an object up an incline (with friction) using a spring-gun. Use energy conservation to find the distance the object travels up the incline.

A block of mass \( m \) is placed in a smooth-bored spring gun at the bottom of the incline so that it compresses the spring by an amount \( x \). The spring has spring constant \( k \). The incline makes an angle \( \theta \) with the horizontal and the coefficient of kinetic friction between the block and the incline is \( \mu \). The block is released, exits the muzzle of the gun, and slides up an incline a total distance \( L \).

**Part A**

Find \( L \), the distance traveled along the incline by the block after it exits the gun. Ignore friction when the block is inside the gun. Also, assume that the uncompressed spring is just at the top of the gun (i.e., the block moves a distance \( x \) while inside of the gun). Use \( g \) for the magnitude of acceleration due to gravity.

**Hint A.1** How to approach the problem

This is an example of a problem that would be very difficult using only Newton's laws and calculus. Instead, use the Work-Energy Theorem: 

\[
E_{\text{final}} = E_{\text{initial}} + W_{\text{net}}
\]

where \( E_{\text{final}} \) is the final energy, \( E_{\text{initial}} \) is the initial energy, and \( W_{\text{net}} \) is the work done on the system by external forces. Let the gravitational potential energy be zero before the spring is released. Then, \( E_{\text{initial}} \) is the potential energy due to the spring, \( E_{\text{final}} \) is the potential energy due to gravity, and \( W_{\text{net}} \) is the work done by friction. Once you've set up this equation completely, solve for \( L \).

**Hint A.2** Find the initial energy of the block

Find the initial energy \( E_{\text{initial}} \) of the block. Take the gravitational potential energy to be zero before the spring is released.

**Hint A.2.1** Potential energy of a compressed spring

Recall that the potential energy \( U \) of a spring with spring constant \( k \) compressed a distance \( x \) is \( U = \frac{1}{2} k x^2 \).

**ANSWER:**

\[
E_{\text{initial}} = \frac{1}{2} k x^2
\]

**Hint A.3** Find the work done by friction

Find \( W_{\text{friction}} \), the work done by friction on the block.

**Hint A.3.1** How to compute work

The work done by a force acting along the direction of motion of an object is equal to the magnitude of the force times the distance over which the object moves. Work is negative if the force directly opposes the motion.

Express \( W_{\text{friction}} \) in terms of \( L \), \( m \), \( g \), \( \mu \), and \( \theta \).

**ANSWER:**

\[
W_{\text{friction}} = -L mg \cos(\theta) \mu
\]

**Hint A.4** Find the final energy of the block
Find an expression for the final energy $E_{\text{final}}$ of the block (the energy when it has traveled a distance $L$ up the incline). Assume that the gravitational potential energy of the block is zero before the spring is released and that the block moves a distance $x_c$ inside of the gun.

**Hint A.4.1 What form does the energy take?**

When the block stops sliding up the ramp, all of its energy is in the form of gravitational potential energy.

Your answer should contain $L$ and $x_c$.

**ANSWER:**

$$E_{\text{final}} = mgy(L + x_c)\sin(\theta)$$

Express the distance $L$ in terms of $x_c$, $k$, $\theta$, $\phi$, and $\mu$.

**ANSWER:**

$$L = \frac{5x_c^2k - mg\sin(\theta)x_c}{mg[\sin(\theta) + \cos(\theta)\mu]}$$

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**Fat: The Fuel of Migrating Birds**

**Description:** Find the maximum distance a bird can fly without feeding, given the amount of fat consumed in the flight and the bird's average speed and power consumption. Also, find how many grams of fat a hummingbird needs to fly across the Gulf of Mexico without feeding.

Small birds can migrate over long distances without feeding, storing energy mostly as fat rather than carbohydrate. Fat is a good form of energy storage because it provides the most energy per unit mass: 1 gram of fat provides about 9.4 (food) Calories, compared to 4.2 (food) Calories per 1 gram of carbohydrate. Remember that Calories associated with food, which are always capitalized, are not exactly the same as calories used in physics or chemistry, even though they have the same name. More specifically, one food Calorie is equal to 1000 calories of mechanical work or 4186 joules. Therefore, in this problem use the conversion factor $1 \text{ Cal} = 4186 \text{ J}$.

**Part A**

Consider a bird that flies at an average speed of 10.7 m/s and releases energy from its body fat reserves at an average rate of 3.70 W (this rate represents the power consumption of the bird). Assume that the bird consumes 4 g of fat to fly over a distance $d_h$ without stopping for feeding. How far will the bird fly before feeding again?

**Hint A.1 How to approach the problem**

From the average speed of the bird, you can calculate how far the bird can fly without stopping if you know the duration of the flight. To determine the duration of the flight, first find the amount of energy available from converting 4 grams of fat, and then use the definition of power.

**Hint A.2 Find the energy used during the flight**

How much energy $E_h$ does the bird have available when it converts 4 grams of fat?

**Hint A.2.1 Converting fat into energy**

As stated in the introduction of this problem, 1 gram of fat provides about 9.4 (food) Calories. Also keep in mind that 1 Calorie = 1000 calories = 4186 J.

Express your answer in kilojoules.

**ANSWER:**

$$E_h = 4.94 \times 4186 \text{ J}$$

**Hint A.3 Find the duration of the flight**

If the bird consumes energy at a rate of 3.70 W, how many hours $t_h$ can it fly using the energy supply provided by 4 grams of fat?

**Hint A.3.1 Definition of power**

The average power $P$ (measured in watts) is the ratio of the energy $\Delta E$ transformed in the time interval $\Delta t$: $P = \frac{\Delta E}{\Delta t}$. Note that power measures either the rate at which energy is transferred (or transformed) or the rate at which work is performed.
Hint A.3.2  
Power: units

Power is measured in watts. One watt is equal to 1 joule per second (i.e., 1 W = 1 J/s).

Express your answer in hours.

\[ t_h = \frac{4.4 \times 10^5}{2} \text{ hr} \]

ANSWER:

\[ t_h = 22000 \text{ hr} \]

Hint A.4  
Find the distance in terms of average velocity

Which of the following expressions gives the distance \( d \) traveled in the time interval \( \Delta t \) at an average speed \( \bar{v} \)?

\[ d = \frac{\Delta x}{\Delta t} \]

ANSWER:

\[ d = \frac{\Delta x}{\Delta t} \]

Now use this expression to find the distance traveled by the bird. Make sure that the units are consistent!

Express your answer in kilometers.

\[ d_h = \frac{4.4 \times 10^5}{3600} \text{ km} \]

ANSWER:

\[ d_h = 4.94 \text{ km} \]

Part B

How many grams of carbohydrate \( m_{\text{carb}} \) would the bird have to consume to travel the same distance \( d_h \)?

Hint B.1  
How to approach the problem

As stated in the introduction of this problem, 1 gram of fat provides about 9.4 Calories, while 1 gram of carbohydrate provides 4.2 Calories.

Express your answer in grams

\[ m_{\text{carb}} = \frac{4.94}{4.2} \text{ g} \]

ANSWER:

\[ m_{\text{carb}} = 1.19 \text{ g} \]

This is more than twice the amount of fat that was needed! In addition, to store 1 gram of carbohydrate (in the form of glycogen, the most common form of animal carbohydrate) about 3 grams of water are needed. Therefore, if energy were stored as carbohydrates, the bird would need to carry more than eight times the fuel mass to perform the same migratory flight!

Part C

Field observations suggest that a migrating ruby-throated hummingbird can fly across the Gulf of Mexico on a nonstop flight traveling a distance of about 800 km. Assuming that the bird has an average speed of 40.0 km/hr and an average power consumption of 1.70 W, how many grams of fat \( m_{\text{fat}} \) does a ruby-throated hummingbird need to accomplish the nonstop flight across the Gulf of Mexico?

Hint C.1  
How to approach the problem

In Part A you were given the amount of fat consumed over the entire flight and were asked to calculate the distance traveled by the migrating bird. Now you need to solve the reverse problem. That is, given the distance traveled, calculate the amount of energy required to perform the flight. Thus, apply the same method as the one used in part A, only in reverse.

From the information on distance and average speed, calculate the duration of the nonstop flight. Then use your result and the given power consumption to determine the amount of energy required for the flight. Finally, calculate how many grams are needed to provide that amount of energy.

Hint C.2  
Find the duration of the flight

How many hours \( t_h \) will the ruby-throated hummingbird fly to travel a distance of 800 km at an average speed of 40.0 km/hr?

Express your answer in hours

\[ t_h = \frac{d_h}{v_h} \text{ hr} \]

ANSWER:

\[ t_h = 22000 \text{ hr} \]
Hint C.3 Find the energy required for the nonstop flight

Given that the hummingbird consumes energy at an average rate of 1.70 \( \text{W} \), how much energy \( E_{\text{hb}} \) will it require during the nonstop flight over the Gulf of Mexico?

**Hint C.3.1 How to use power and time**

Remember that power is energy transferred (or transformed) per unit time. Make sure you are using the correct unit for time.

Express your answer in joules.

\[
E_{\text{hb}} = \frac{P_{\text{hb}} \cdot t_{\text{hb}}}{43600} \text{ J}
\]

Now calculate how many grams of fat are required to provide this amount of energy. Recall that 1 gram of fat provides 9.4 Calories of energy.

Express your answer in grams.

\[
m_{\text{fat}} = \frac{E_{\text{hb}}}{9.4 \cdot (186)} \text{ g}
\]

Considering that in normal conditions the mass of a ruby-throated hummingbird is only 3 or 4 grams, the bird will need to almost double its body mass to store enough fat to perform the nonstop flight.

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**Problem 11.11**

**Description:** The two ropes seen in the figure are used to lower a 255 kg piano from a second-story window to the ground. (a) How much work is done by each of the three forces?

The two ropes seen in the figure are used to lower a 255 kg piano 6.30 m from a second-story window to the ground.

**Part A**

How much work is done by each of the three forces?

Enter your answers numerically separated by commas.

\[
W_G, W_1, W_2 = \frac{-1830 \text{N} \cos(30^\circ)}{\sqrt{3}} \text{ J}, \frac{-1295 \text{N} \cos(45^\circ)}{\sqrt{2}} \text{ J}
\]

---

**Problem 11.68**

**Description:** A Porsche 944 Turbo has a rated engine power of 217 hp. 30% of the power is lost in the drive train, and 70% reaches the wheels. The total mass of the car and driver is \( m \), and two-thirds of the weight is over the drive wheels. (a) What is the maximum …

A Porsche 944 Turbo has a rated engine power of 217 hp. 30% of the power is lost in the drive train, and 70% reaches the wheels. The total mass of the car and driver is 1500 kg, and two-thirds of the weight is over the drive wheels.

**Part A**

What is the maximum acceleration of the Porsche on a concrete surface where \( \mu_s = 1.00 \)?

**Hint A.1**

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What force pushes the car forward?

**ANSWER:**

\[
\alpha_{\text{max}} = 6.53 \text{ m/s}^2
\]

**Part B**

What is the speed of the Porsche at maximum power output?

**ANSWER:**

\[
\nu = \frac{217.76}{\text{rpm}} \cdot 0.7 \text{ m/s}
\]

**Part C**

If the Porsche accelerates at \( \alpha_{\text{max}} \), how long does it take until it reaches the maximum power output?

**ANSWER:**

\[
\tau = \frac{217.76}{\text{rpm}} \cdot 0.7 \text{ s}
\]

---

**Visualizing Rotation**

**Description:** Select the rotating disks in the movie that rotate with specific properties (e.g., constant negative angular acceleration). Uses applets.

**Learning Goal:** To be able to identify situations with constant angular velocity or constant angular acceleration by watching movies of the rotations.

Recall that angular velocity measures the angle through which an object turns over time. If a disk has constant angular velocity and it makes a quarter revolution in one second, then it will make another quarter revolution the next second. If the disk turns in the clockwise direction, it has, by definition, negative angular velocity. The magnitude of the angular velocity is the angular speed. This applet shows a few rotating disks and lists their angular velocities, should help you to get a feel for how different angular velocities look.

Angular acceleration measures how the angular velocity changes over time. If a disk has constant angular velocity, then it has zero angular acceleration. If a disk turns a quarter revolution one second and a half revolution the next second, then its angular velocity is changing, and so it has an angular acceleration. This applet shows two disks and lists their initial angular velocities and angular accelerations. This should help you to get a feel for how different angular accelerations look. Just as with linear accelerations, if a positive angular velocity decreases, that indicates a negative angular acceleration. If a negative angular velocity becomes more negative (i.e., its magnitude increases), that also indicates a negative angular acceleration.

This applet shows six disks rotating with constant angular acceleration. No two have the same initial angular velocity and angular acceleration. To answer the following questions, number the disks starting from the top. That is, call the yellow disk "1" and go sequentially down to the red disk, which will be "6". In the following questions, you will be asked to determine whether the disks' angular velocities and accelerations are positive, negative, or zero. Keep in mind that angular velocity is considered positive if rotation is in the counterclockwise direction. Angular acceleration is positive if the rotation is in the counterclockwise (positive) direction and the angular speed is increasing, or if rotation is the clockwise (negative) direction and the angular speed is decreasing (thus the angular velocity is becoming less negative). Negative angular acceleration is defined analogously.

**Part A**

Which of the disks have positive initial angular velocity?

Write down the numbers, in order, that correspond to the disk(s) that you believe are correct, without commas or spaces between them. For example, if you think that the yellow disk and the gray disk are the correct ones, then you should enter 15.

**ANSWER:** 46

---

**Part B**

Which of the disks have negative initial angular velocity?

Write down the numbers, in order, that correspond to the disk(s) that you believe are correct, without commas or spaces between them. For example, if you think that the yellow disk and the gray disk are the correct ones, then you should enter 15.

**ANSWER:** 25

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**Part C**

Which of the disks have zero angular acceleration?

Write down the numbers, in order, that correspond to the disk(s) that you believe are correct, without commas or spaces between them. For example, if you think that the
yellow disk and the gray disk are the correct ones, then you should enter 15.

ANSWER: 56

Part D
Which of the disks have positive angular acceleration?
Write down the numbers, in order, that correspond to the disk(s) that you believe are correct, without commas or spaces between them. For example, if you think that the yellow disk and the gray disk are the correct ones, then you should enter 15.

ANSWER: 12

Part E
Which of the following characterizes the initial angular velocity $\omega_0$ and the angular acceleration $\alpha$ of disk 4?

ANSWER:
- $\omega_0 > 0$ and $\alpha > 0$
- $\omega_0 = 0$ and $\alpha > 0$
- $\omega_0 < 0$ and $\alpha > 0$
- $\omega_0 > 0$ and $\alpha = 0$
- $\omega_0 > 0$ and $\alpha < 0$
- $\omega_0 = 0$ and $\alpha = 0$

Weight and Wheel

Description: A block is attached (via a string) to a wheel at two different radii. Determine whether the final angular velocity of the wheel after the block falls a given height is greater for the longer or shorter radius.

Consider a bicycle wheel that initially is not rotating. A block of mass $m_B$ is attached to the wheel and is allowed to fall a distance $h$. Assume that the wheel has a moment of inertia $I$ about its rotation axis.

Part A
Consider the case that the string tied to the block is attached to the outside of the wheel, at a radius $r_A$.

Find $\omega_{Af}$, the angular speed of the wheel after the block has fallen a distance $h$, for this case.

Hint A.1 How to approach this problem
The most straightforward way to solve this problem is to use conservation of mechanical energy. The total initial energy of the system is equal to the total final energy of the system (where the system consists of the wheel and the block). In other words,

$$E_i = E_{\text{fin}} + E_{\text{pot}}$$

Where $E_i$ is the initial energy of the system, $E_{\text{fin}}$ is the final energy of the block and $E_{\text{pot}}$ is the final energy of the wheel.

Hint A.2 Initial energy of the system
Initially, the wheel is not rotating. The initial energy of the system consists of the gravitational potential energy stored in the block, since it is not moving either. Supposing that the gravitational potential energy of the block is zero at "ground level," find the initial energy of the system.

ANSWER:

$$E_i = mgh$$
What about the potential energy for the wheel? If you were to be very particular, you would either need to assign some potential energy for the wheel, or choose a different reference height for it (located at it's center), such that it's potential energy was zero. However, since you know that the (center of mass of the) wheel does not move, it's potential energy does not change and as such you don't really need to calculate it. Again, this is because it is the changes in potential energy and not the absolute value of potential energy that are important.

**Hint A.3** Final energy of block

Find the final energy of the block.

**Hint A.3.1** Final velocity of the block

Find \( v_f \), the magnitude of the final velocity of the block.

Express the velocity in terms of \( r_A \) and the final angular velocity of the wheel, \( \omega_f \).

**ANSWER:**

\[ v_f = r_A \omega_f \]

Express the final energy of the block in terms of given quantities (excluding \( h_f \)) and the unknown final angular velocity of the wheel, \( \omega_f \).

**ANSWER:**

\[ E_{\text{fin}} = \frac{1}{2} m (r_A \omega_f)^2 \]

Since we are measuring potentials w.r.t. the ground for the block, the final potential energy of the block is zero.

**Hint A.4** Final energy of wheel

Find the final kinetic energy of the wheel.

Express your answer in terms of \( I \) (the wheel’s moment of inertia) and \( \omega_f \).

**ANSWER:**

\[ E_{\text{fin}} = \frac{1}{2} I \omega_f^2 \]

Express \( \omega_f \) in terms of \( m \), \( g \), \( h \), \( r_A \), \( I \).

**ANSWER:**

\[ \omega_f = \sqrt{\frac{2mgh}{mr_A^2 + I}} \]

**Part B**

Now consider the case that the string tied to the block is wrapped around a smaller inside axle of the wheel of radius \( r_B \).

Find \( \omega_B \), the angular speed of the wheel after the block has fallen a distance \( h \), for this case.

**Hint B.1** Similarity to previous part

The derivation of \( \omega_B \) is exactly the same as the derivation for \( \omega_A \), using \( r_B \) instead of \( r_A \).

Express \( \omega_B \) in terms of \( m \), \( g \), \( h \), \( r_B \), and \( I \).

**ANSWER:**

\[ \omega_B = \sqrt{\frac{2mgh}{mr_B^2 + I}} \]

**Part C**
Which of the following describes the relationship between \( \omega_A \) and \( \omega_B \)?

**Hint C.1 How to approach this question**

To figure out which angular velocity is greater (\( \omega_A \) or \( \omega_B \)), you only need to consider the radius dependence of the expression for \( \omega \). Ignoring all of the other parameters, you should have found that \( \omega \) goes as \( 1/\text{radius} \) (where “radius” refers to where the string is attached, which is not necessarily the outer radius of the wheel). The problem then reduces to figuring out which is greater, \( 1/r_A \) or \( 1/r_B \).

**ANSWER:**

- \( \omega_A > \omega_B \)
- \( \omega_B > \omega_A \)
- \( \omega_A = \omega_B \)

This is related to why gears are found on the inside rather than on the outside of a wheel.

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**Problem 12.8**

**Description:** A \( m_1 \) ball and a \( m_2 \) ball are connected by a \( l \)-cm-long, massless, rigid rod. The balls rotate about their center of mass at \( \omega \). (a) What is the speed of the \( m_1 \) ball?

A 150 g ball and a 210 g ball are connected by a 31-cm-long, massless, rigid rod. The balls rotate about their center of mass at 130 rpm.

**Part A**

What is the speed of the 150 g ball?

Express your answer using two significant figures.

**ANSWER:**

\[ v = \frac{m_1 \omega}{m_1 + m_2} \text{ m/s} \]

---

**Power Dissipation Puts a Drag on Racing**

**Description:** Compute the change in top speed of a car, subject to form drag, when the maximum power of the car's engine is increased by 10%.

The dominant form of drag experienced by vehicles (bikes, cars, planes, etc.) at operating speeds is called form drag. It increases quadratically with velocity (essentially because the amount of air you run into increases with \( v \) and so does the amount of force you must exert on each small volume of air). Thus

\[ P_{\text{drag}} = C_d A v^2, \]

where \( A \) is the cross-sectional area of the vehicle and \( C_d \) is called the coefficient of drag.

**Part A**

Consider a vehicle moving with constant velocity \( \vec{v} \). Find the power dissipated by form drag.

**Hint A.1 How to approach the problem**

Because the velocity of the car is constant, the drag force is also constant. Therefore, you can use the result that the power \( P \) provided by a constant force \( \vec{F} \) to an object moving with constant velocity \( \vec{v} \) is \( P = \vec{F} \cdot \vec{v} \). Be careful to consider the relative direction of the drag force and the velocity.

Express your answer in terms of \( C_d, A, \) and speed \( v \).

**ANSWER:**

\[ P = -C_d A v^2 \]

---

**Part B**

A certain car has an engine that provides a maximum power \( P_b \). Suppose that the maximum speed of the car, \( v_b \), is limited by a drag force proportional to the square of the speed (as in the previous part). The car engine is now modified, so that the new power \( P_1 \) is 10 percent greater than the original power (\( P_1 = 1.10 P_b \)).

Assume the following:

- The top speed is limited by air drag.
- The magnitude of the force of air drag at these speeds is proportional to the square of the speed.

By what percentage, \( (v_1 - v_b)/v_b \), is the top speed of the car increased?
Hint B.1  
Find the relationship between speed and power

If the magnitude of the air-drag force is proportional to the square of the car's speed, how is the power delivered, $P$, related to the speed $v$?

**ANSWER:**
- $P \propto v$
- $P \propto v^2$
- $P \propto v^3$

Hint B.2  
How is the algebra done?

The relationship between the new power and the old power is $P_1 = 1.3 P_0$. The relationship between the new top speed and the old top speed can be written as $v_f = (1 + a) v_0$, where $a$ is the percent change in top speed. Finally, power $P$ is related to maximum speed $v$ by the formula $P \propto v^n$.

What is $a$ in terms of $n$?

**Hint B.2.1**  
Help with some math

Starting with the relationship

$$P_1 \propto v_1^n$$

Substitute in the expressions for $P_1$ and $v_1$ in terms of $P_0$ and $v_0$.

$$1.3 P_0 \propto (1 + a) v_0^n$$

Then, divide this last expression by the relationship

$$P_0 \propto v_0^n$$

This is a general approach to scaling problems. The advantage is that the unknown constant of proportionality (in this case $C_d \Delta A$) divides out.

**ANSWER:**
- $1.1^{1.6} - 1$
- $1.1^1$
- $1.1^{0.8}$
- $1.1^{0.6}$

Express the percent increase in top speed numerically to two significant figures.

**ANSWER:**
- $8.2$
- $(v_1 - v_0)/v_0 = 0.032 \%$

You'll note that your answer is very close to one-third of the percentage by which the power was increased. This dependence of small changes on each other, when the quantities are related by proportionalities of exponents, is common in physics and often makes a useful shortcut for estimations.

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**A Tale of Two Nutcrackers**

**Description:** Static torque problem which requires the balancing of torques to show how a force can be amplified by the ratio of the moment arm of a lever.

This problem explores the ways that torque can be used in everyday life.

**Case 1**

To crack a nut a force of magnitude $F_n$ (or greater) must be applied on both sides, as shown in the figure. One can see that a nutcracker only applies this force at the point in which it contacts the nut (at a distance $d$ from the nutcracker pivot). In this problem the nut is placed in a nutcracker and equal forces of magnitude $F$ are applied to each end, directed perpendicular to the handle, at a distance $D$ from the pivot. The frictional forces between the nut and the nutcracker are equal and large enough that the nut doesn’t shoot out of the nutcracker.
Part A

Find $F$, the magnitude of the force applied to each side of the nutcracker required to crack the nut.

**Hint A.1 Sum of torques**

There is no angular acceleration. What will the sum of the torques be about any point?

**Hint A.2 Sum of torques about a specific point**

Take the hinge as the pivot point. Note that the forces at the pivot point give no torque. Since the forces are symmetric about the $x$ axis we can just consider the top piece of the nutcracker. What is the sum of the torques about the hinge $\Sigma \tau$ generated by forces on the top piece (both $F$ and $F_0$)? Consider counterclockwise (right-handed) torques to be positive.

*Answer in terms of given quantities.*

**Answer:**

$$\Sigma \tau = -FD + F_0d$$

Express the force in terms of $F_0$, $d$, and $D$.

**Answer:**

$$F = \frac{F_0d}{D}$$

Case 2

The nut is now placed in a nutcracker with only one lever, as shown, and again friction keeps the nut from slipping. The top "jaw" (in black) is fixed to a stationary frame so that a person just has to apply a force to the bottom lever. Assume that $F_0$ is directed perpendicular to the handle.

Part B

Find the magnitude of the force $F_2$ required to crack the nut.

**Hint B.1 Sum of torques**

Again, the sum of the torques about the hinge will be zero. Also, taking the hinge as a pivot point will again eliminate torques due to the forces between the two pieces of the nutcracker.

*Express your answer in terms of $F_0$, $d$, and $D$.*

**Answer:**

$$F_2 = \frac{F_0d}{D}$$

Part C

For the first nutcracker, two applied forces of magnitude $F$ were required to crack the nut, whereas for the second, only one applied force of magnitude $F$ was required. How would you explain this difference?

**Answer:**

In the second case the nutcracker handle is effectively longer and generates twice the torque of that in the first case.
There is an additional force of magnitude $F_2$ applied to the nut by the fixed jaw in the second case. This jaw is held fixed by external forces (such as the forces due to the frame). The net torque on the nutcracker in the second case is non-zero.

### A Rolling Hollow Sphere

**Description:** For a hollow spherical shell rolling down a slope without slipping, calculate its acceleration, the frictional force, and the minimum coefficient of friction to prevent slipping.

A hollow spherical shell with mass 1.90 kg rolls without slipping down a slope that makes an angle of 40.0° with the horizontal.

#### Part A

Find the magnitude of the acceleration $a_{cm}$ of the center of mass of the spherical shell.

**Hint A.1 How to approach the problem**

First, draw a diagram of the system, including all forces acting on the sphere. Assume that the positive $x$ direction points downward along the slope and the positive $y$ direction points upward normal to the slope, making the positive $z$ direction out of the screen. The angular speed and angular acceleration of the spherical shell are negative (clockwise) around the $z$ axis. Using this coordinate system, determine the components of all forces in the $x$ and $y$ directions, and set up the corresponding Newtonian equations for the translational and rotational motions of the shell. Since there is no slipping, use both equations to calculate the acceleration by solving the angular motion equation for the frictional force in terms of the translational acceleration and then substituting into the translational motion equation.

**Hint A.2 Translational motion in the $x$ direction**

In the $x$ direction, the only forces that are acting on the spherical shell are the frictional force, pointing up the slope, and the component of the weight that points down the slope. Be careful to use the correct trigonometric function (sine or cosine) when finding the $x$ component of the weight of the shell.

**Hint A.3 Torque on the spherical shell**

Since the only force acting away from the center of the spherical shell is the friction, this will cause the angular acceleration of the sphere by creating a torque of $\tau = fR$, where $R$ is the radius of the spherical shell.

**Hint A.4 Moment of inertia**

The moment of inertia of a spherical shell is $I = \frac{2}{3}mR^2$, where $R$ is the radius of the spherical shell.

**Hint A.5 Relation between the translational and angular accelerations**

Since the spherical shell rolls down the slope without slipping, the translational and rotational speeds of the shell must cancel each other at the surface of the slope, giving $v_{cm} = R\omega$, where $R$ is the radius of the spherical shell. If the derivative is taken with respect to time, it becomes evident that $a_{cm} = R\alpha$.

Take the free-fall acceleration to be $g = 9.80 \text{ m/s}^2$.

**ANSWER:**

\[ a_{cm} = \frac{2}{3}m \sin(\theta) \text{ m/s}^2 \]

#### Part B

Find the magnitude of the frictional force acting on the spherical shell.

**Hint B.1 How to approach the problem**

As in Part A, you should first draw a diagram of the system, including all forces acting on the sphere. Assume that the positive $x$ direction points downward along the slope and that the positive $y$ direction points upward normal to the slope, making the positive $z$ direction out of the screen. The angular speed and angular acceleration of the spherical shell are negative (clockwise) around the $z$ axis. Using this coordinate system, determine the components of all forces in the $x$ and $y$ directions, and set up the corresponding Newtonian equations for the translational and rotational motions of the shell. Since there is no slipping, use both equations together to calculate the acceleration by solving the angular motion equation for the frictional force in terms of the translational acceleration, and then substituting into the translational motion equation.

Take the free-fall acceleration to be $g = 9.80 \text{ m/s}^2$.

**ANSWER:**

\[ f = \frac{2}{3}m \sin(\theta) \text{ N} \]

The frictional force keeps the spherical shell stuck to the surface of the slope, so that there is no slipping as it rolls down. If there were no friction, the shell would simply slide down the slope, as a rectangular box might do on an inclined (frictionless) surface.

#### Part C

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Find the minimum coefficient of friction $\mu$ needed to prevent the spherical shell from slipping as it rolls down the slope.

**Hint C.1 How to approach the problem**

Since the frictional force was calculated in the previous part, use the equation relating the frictional force to the normal force. Be careful about the trigonometric functions used to solve for the coefficient of friction.

**ANSWER:**

$$\mu = \frac{2}{5} \tan(\theta)$$

---

**Problem 12.25**

**Description:** An object’s moment of inertia is $I$. Its angular velocity is increasing at the rate of $\alpha$. (a) What is the torque on the object?

An object’s moment of inertia is 2.00 kg m$^2$. Its angular velocity is increasing at the rate of 3.00 rad/s$^2$.

**Part A**

What is the torque on the object?

**ANSWER:**

$\tau = \alpha \frac{I}{S}$ Nm

---

**Problem 12.74**

**Description:** A hollow sphere is rolling along a horizontal floor at $v$ when it comes to a $\alpha$ incline. (a) How far up the incline does it roll before reversing direction?

A hollow sphere is rolling along a horizontal floor at 7.00 m/s when it comes to a 37.0° incline.

**Part A**

How far up the incline does it roll before reversing direction?

**ANSWER:**

$$\frac{\tan(\theta)}{\sin(\theta)} \ m$$

---

**Problem 12.83**

**Description:** A $m_b$ wood block hangs from the bottom of a $m_r, 1$-m-long rod. The block and rod form a pendulum that swings on a frictionless pivot at the top end of the rod. A $m$ bullet is fired into the block, where it sticks, causing the pendulum to swing out to a ...

A 2.3 kg wood block hangs from the bottom of a 1.3 kg, 1.0-m-long rod. The block and rod form a pendulum that swings on a frictionless pivot at the top end of the rod. A 14 g bullet is fired into the block, where it sticks, causing the pendulum to swing out to a 33° angle.

**Part A**

What was the speed of the bullet? You can treat the wood block as a particle.

Express your answer using two significant figures.

**ANSWER:**

$$v = \sqrt{\frac{2}{\mu} \left( \frac{1}{\mu^2} \right) \left( \frac{2m_b + 2m + m_r}{\mu} \right) \frac{1}{2} (1 - \cos(\theta))} \ m/s$$

---

**Work Raising an Elevator**

**Description:** Use readouts of work and energy in an applet of an elevator being raised to determine the mass of the elevator and the tension in its cable. (uses applet)

Look at this applet. It shows an elevator with a small initial upward velocity being raised by a cable. The tension in the cable is constant. The energy bar graphs are marked in intervals of 600 J.

**Part A**
What is the mass of the elevator? Use \( g = 10 \text{ m/s}^2 \) for the magnitude of the acceleration of gravity.

**Hint A.1** Using the graphs
Think about which graph(s) show energies that are directly related to the mass of the elevator. There may be more than one. You would like to get the most accurate number you can, so choose the graph that you can read most accurately.

**Hint A.2** Needed formula
Recall that the gravitational potential energy near the earth's surface is given by \( U = mgh \), where \( m \) is the mass of the object, \( g \) is the magnitude of the gravitational acceleration, and \( h \) is the height above the ground.

Express your answer in kilograms to two significant figures.

**ANSWER:**
\[ m = 60 \text{ kg} \]

---

**Part B**
Find the magnitude of the tension in the cable. Be certain that the method you are using will be accurate to two significant figures.

**Hint B.1** How to approach the problem
In the previous part, you could use the graph of potential energy to determine the mass to two significant figures, because when the elevator stopped, the top of the potential energy bar lay right on one of the grid lines. In this problem, you could use the graph of work to find the tension, but since it lies somewhere between the grid lines, it is unlikely that you could determine the tension to the necessary accuracy. However, it is a good way to get an estimate with which to check your answer.

The numerical data given in the window beneath the graphs do have two significant figures of accuracy, and thus they could be used in combination with the data in the graph of the final energy to get a more accurate value for the work done on the elevator. Recall, in fact, that the work done on the elevator by the tension must equal the change in mechanical energy of the system.

**Hint B.2** Find the change in mechanical energy
From the information given in the applet and the information found in Part A, determine the change in the total mechanical energy of the system \( \Delta E \).

**Hint B.2.1** Find the initial mechanical energy
Assuming that the potential energy of the elevator at the instant when you run the simulation is zero, what is the initial mechanical energy \( E_{\text{initial}} \) of the system?

**Hint B.2.1.1** Definition of mechanical energy
Recall that the mechanical energy of a system is defined as the sum of kinetic energy and potential energy,
\[ E = K + U \]

Note that, at the instant when you run the simulation, the potential energy \( U \) of the elevator is zero. Thus, the total initial mechanical energy of the system is simply given by the initial kinetic energy of the elevator \( K = \frac{1}{2}mv^2 \), which can be evaluated from the information about the mass of the elevator found in Part A, and the information about the initial speed of the elevator given in the window beneath the bar graphs in the applet.

Express your answer in joules to two significant figures.

**ANSWER:**
\[ E_{\text{initial}} = 480 \text{ J} \]

The total mechanical energy of the system can be determined from the data in the energy bar graphs given in the applet, just as you did in Part A to find the mass of the elevator.

Express your answer in joules to two significant figures.

**ANSWER:**
\[ \Delta E = 1900 \text{ J} \]

Since the change in the total mechanical energy of the system must equal the work \( W \) done by the tension, your answer gives a more accurate estimate of \( W \) than what you could have calculated from the data in the work bar graphs in the applet. Now use the information about the distance moved by the elevator given in the window beneath the graphs to find the tension.

Express your answer in newtons to two significant figures.

**ANSWER:**
\[ T = 480 \text{ N} \]
Hill's Law Conceptual Question

**Description:** Conceptual question on the relationship between force, velocity, and power.

Imagine that you're loading a pickup truck with bags of groceries. You notice that the smaller the weight you attempt to lift, the quicker you can lift it. However, you also notice that there is a limit to how quickly you can lift even very small weights, and that above a certain weight, you can no longer lift the weight at all. The detailed relationship between the contraction velocity of a muscle (the speed with which you can lift something) and the weight you are attempting to lift, is known as Hill’s law.

**Part A**

Based on this description, which of the following graphs of velocity vs. force is a possible representation of Hill’s law?

**Hint A.1 Maximum weight**

Hill’s law states that there exists a maximum force that a muscle can exert, and thus a maximum weight that a muscle can lift. Only one graph has a limit to the maximum force that can be produced.

**Answer:**

- A
- B
- C
- D
- E

**Part B**

The power exerted by a muscle is the product of the force exerted and the velocity of contraction. The area of which of these shaded regions represents the power exerted while a weight is lifted at maximum speed?

**Hint B.1 How to approach the problem**

In this example, power is the product of the force exerted by the muscles and the contraction velocity of the muscles. In lifting any given weight, our muscles have a single maximum contracting velocity; we assume, in this case, that the weight is lifted at this velocity. In lifting a given weight at a constant velocity, your muscles exert a constant force. Therefore, in this example, both the force and the contraction velocity are constant for a given weight. How can the product of two constant numbers be shown graphically?

**Answer:**

- A
- B
- C
- D
- None of the above

The power produced by a muscle is represented by the area of the rectangle formed by the two coordinate axes and the point on the Hill’s law graph representing the weight being lifted. Notice that if you lift a very large weight (near the limit of the maximum force your muscle can produce), the area of this "long and skinny" rectangle can be quite small. If you lift a very small weight, the area of this "tall and skinny" rectangle can also be quite small. However, if you lift a weight near the middle of your weightlifting range, the area of the rectangle, and hence the power produced by your muscle, is a maximum.
Balancing seesaw: intuition, how it relates to torque, and finally balancing by using horizontal force.

Learning Goal: To make the connection between your intuitive understanding of a seesaw and the standard formalism for torque.

This problem deals with the concept of torque, the “twist” that an off-center force applies to a body that tends to make it rotate.

Try to use your intuition to answer the following question. If your intuition fails, work the rest of the problem and return here when you feel that you are more comfortable with torques.

Part A

A mother is helping her children, of unequal weight, to balance on a seesaw so that they will be able to make it tilt back and forth without the heavier child simply sinking to the ground. Given that her heavier child of weight $W$ is sitting a distance $L_1$ to the left of the pivot, at what distance $L_2$ must she place her second child of weight $W_2$ on the right side of the pivot to balance the seesaw?

Hint A.1 How to approach the problem

Consider whether $L_2$ increases or decreases as each of the variables that it depends on-- $W$, $L_1$, and $L_2$--is made larger or smaller.

Express your answer in terms of $L_1$, $W$, and $L_2$.

ANSWER: $L_2 = \frac{L_1 W}{W}$

Try to use your intuition to answer the following question. If your intuition fails, work the rest of the problem and return here when you feel that you are more comfortable with torques.

Part B

Find $\tau_{p,w}$, the torque about the pivot due to the weight $W$ of the smaller child on the seesaw.

Express your answer in terms of $L_1$ and $W$.

ANSWER: $\tau_{p,w} = -L_1 W$

Part C

Determine $\sum \tau_{p,q}$, the sum of the torques on the seesaw. Consider only the torques exerted by the children.
Hint C.1  Torque from the weight of the seesaw

The seesaw is symmetric about the pivot, and so the gravitational force on the seesaw produces no net torque. More generally, when determining torques, the gravitational force on an object in a uniform gravitational field can be taken to act at the center of mass. Here the center of mass is directly above the pivot, so the weight of the seesaw has zero moment arm and produces no torque about the pivot.

Express your answer in terms of $W$, $t$; $L$ and $L_T$

**ANSWER:** $\sum \tau_i = 0$  \( WL - wL_t \)

Good! If you did not solve for the distance $L_3$ required to balance the seesaw in Part A, do so now.

The equation $\sum \tau_i = 0$ applies to any body that is not rotationally accelerating. Combining this equation with $\sum F_x = 0$ (which applies to any body that is not accelerating linearly) gives a pair of equations that are sufficient to form the basis of statics; these or similar, often more complicated equations govern structures that are not accelerating. The torque equation is often of more utility, however, because you can choose the pivot point arbitrarily (often so that unknown forces have no moment arm and therefore contribute no torque). The art of applying these equations to large or complicated structures constitutes a significant part of mechanical and civil engineering.

The child with weight $w$ has an identical twin also of weight $w$. The two twins now sit on the same side of the seesaw, with one twin a distance $L_3$ from the pivot and the other a distance $L_2$.

Part D
Where should the mother position the child of weight $W$ to balance the seesaw now?

Hint D.1  Balancing the seesaw

For the seesaw to balance, the sum of the torques from the three children $\sum \tau_i$ must be zero. What is the sum of the torques?

Express your answer in terms of $L_1$, $L_2$, $L_3$, $W$, and $w$.

**ANSWER:** $\sum \tau_i = 0$  \( LW - w(L_3 + L_2) \)

Now solve your torque equation for $L$

Express your answer in terms of $L_3$, $L_2$, $W$, and $w$.

**ANSWER:**  \( L = \frac{w(L_3 + L_2)}{W} \)

Bad news! When the mother finds the distance $L$ from the previous part it turns out to be greater than $L_{end}$, the distance from the pivot to the end of the seesaw. Hence, even with the child of weight $W$ at the very end of the seesaw the twins exert more torque than the heavier child does. The mother now elects to balance the seesaw by pushing sideways on an ornament (shown in red) that is a height $h_1$ above the pivot.

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Part E

With what force in the rightwards direction, \( F_{\text{m}} \), should the mother push? Note that if you think the force exerted by the mother should be toward the left, your final answer \( F_{\text{m}} \) should be negative.

**Hint E.1 Sign conventions**

It is easy to make sign errors in torque problems, and experience shows that it is better to use standard conventions (here that \( F_{\text{m}} \) goes to the right, the direction of positive \( x \)-component) than to change the direction of positive torque or displacement to suit your convenience. (You are likely to forget your unconventional choice at a later point in the problem.) In this case, your intuition correctly expects that the mother must push to the left to make things balance; the equations will confirm this by giving a negative result for \( F_{\text{m}} \). (A positive result would mean that the force would be directed to the right in the figure.)

**Hint E.2 Torque due to mother’s push**

The sum of all four torques (due to each of the three children plus the mother) must be zero. What is the torque \( \tau_{\text{m}} \) due to the mother’s push?

Remember, a positive torque will cause counterclockwise rotation of the seesaw.

Express your answer in terms of the unknown force \( F_{\text{m}} \) and the height at which it is applied \( h_{\text{m}} \).

**Answer:**

\[ \tau_{\text{m}} = -F_{\text{m}}h \]

Express your answer in terms of \( W, I_{\text{end}} \), \( L_{1}, L_{2}, L_{3} \), and \( h_{\text{m}} \).

**Answer:**

\[ F_{\text{m}} = \frac{W(L_{1} - w(L_{2} + L_{3}))}{h_{\text{m}}} \]

This answer will necessarily be negative, because you were told that to balance the seesaw with the twins on the right, the child of weight \( W \) had to be “beyond the end” of the seesaw. Therefore, when \( L_{\text{end}} \) is the position of the child of weight \( W \), \( F_{\text{m}} \) will be less than zero. Hence the mother must push to the left as you’d expect.

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**A Sculpture in Equilibrium**

**Description:** Locate the center of mass (center of gravity) of an irregularly shaped, balanced object.

The abstract sculpture shown in the figure can be placed on a horizontal surface without tipping over. Different parts of the sculpture may be hollow or solid, or made of different materials, but the artist isn’t revealing any such information.

---

**Part A**

Five locations labeled A through E are indicated on the diagram. Which of these, if any, is a possible location of the object’s center of mass (center of gravity)?

**Hint A.1 The torque(s) on the sculpture**

The sculpture is in equilibrium. This means that the net torque that acts on the sculpture is zero.

The only forces that act on the sculpture are the normal force acting on the “foot” of the sculpture and the weight acting effectively at the center of mass. If you use the foot as the axis of rotation, the weight supplies the only torque on the sculpture. Think about where the weight must be located in order for this torque to be zero and the sculpture to be in equilibrium.

**Answer:**

\[ \text{B} \]
For an object to be in equilibrium under the action of its own weight, the center of mass of the object must lie directly above the base of the object. This ensures that the torque due to the weight of the object will be zero about an axis passing through a point in the base. The objects in the figure have their centers of mass marked by a dot. The left object is in equilibrium, since the center of mass is above a point on its base. The right object, however, will tip over clockwise, since its weight supplies an unbalanced clockwise torque about any point on its base.

Torque Magnitude Ranking Task

Description: Rank the torques induced by various forces acting on a wrench. (ranking task)

The wrench in the figure has six forces of equal magnitude acting on it.

Part A

Rank these forces (A through F) on the basis of the magnitude of the torque they apply to the wrench, measured about an axis centered on the bolt.

Hint A.1  Definition of torque

Torque is a measure of the "twist" that an applied force exerts on an object. Mathematically, torque is defined as

\[ \tau = rF \sin \theta, \]

where \( r \) is the magnitude of the displacement vector from the rotation axis to the point of application of the force of magnitude \( F \), and \( \theta \) is the angle between this displacement and the applied force, as shown in the figure.

The direction of a torque can be either counterclockwise (as above) or clockwise. This is determined by the direction the object will rotate under the action of the force.

Hint A.2  Maximum torque

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Based on the mathematical definition of torque, torque is maximized when the force is large in magnitude, located a large distance from the axis of interest, and oriented perpendicular to the displacement \( \vec{r} \), which is often referred to as the lever arm of the force.

Rank from largest to smallest. To rank items as equivalent, overlap them.

**ANSWER:**

**View**

---

**Pivoted Rod with Unequal Masses**

**Description:** Find the moment of inertia for a rod with two masses (treated as point particles) attached to its ends. Then compute the angular acceleration of the rod when it is released from rest. Then find the torque and initial angular acceleration about the pivot point.

The figure shows a simple model of a seesaw. These consist of a plank/rod of mass \( m_0 \) and length \( 2L \) allowed to pivot freely about its center (or central axis), as shown in the diagram. A small sphere of mass \( m_1 \) is attached to the left end of the rod, and a small sphere of mass \( m_2 \) is attached to the right end. The spheres are small enough that they can be considered point particles. The gravitational force acts downward. The magnitude of the acceleration due to gravity is equal to \( g \).

**Part A**

What is the moment of inertia \( I \) of this assembly about the axis through which it is pivoted?

**Hint A.1 How to approach the problem**

The moment of inertia of the assembly about the pivot is equal to the sum of the moments of inertia of each of the components of the assembly about the pivot point. That is, the total moment of inertia is equal to the moment of inertia of the rod plus the moment of inertia of the particle of mass \( m_1 \), plus the moment of inertia of the particle of mass \( m_2 \), all measured with respect to the pivot point.

**Hint A.2 Find the moment of inertia due to the sphere of mass \( m_1 \)**

What is the moment of inertia of the particle of mass \( m_1 \) measured about the pivot point?

**Hint A.2.1 Formula for moment of inertia**

Consider an object consisting of particles with masses \( m_i \). Let \( r_i \) be the distance of the \( i \)th particle from the axis of rotation. Then the moment of inertia \( I \) of the object about the axis of rotation is given by

\[
I = \sum_i m_i r_i^2
\]

Express your answer in terms of given quantities.

**ANSWER:**

\[
I_1 = m_1 x_1^2
\]

**Hint A.3 Find the moment of inertia due to the sphere of mass \( m_2 \)**

What is the moment of inertia of the particle of mass \( m_2 \) measured about the pivot point?

Express your answer in terms of given quantities.

**ANSWER:**

\[
I_2 = m_2 x_2^2
\]

**Hint A.4 Find the moment of inertia of the rod**

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What is the moment of inertia of the rod about the pivot point?

**Hint A.4.1 General formula for the moment of inertia of a rod**

Consider a rod of total length \( L \) and mass \( m \), pivoted about its center. (In this problem, \( L \) equals \( 2R \).) What is the moment of inertia of the rod about its pivot point?

**ANSWER:**

Express \( I \) in terms of \( mL \) and \( x \).

**ANSWER:**

Express the moment of inertia in terms of \( mL, m_1, m_2, \) and \( x \). Keep in mind that the length of the rod is \( 2R \), not \( x \).

**ANSWER:**

Part B

Suppose that the rod is held at rest horizontally and then released. (Throughout the remainder of this problem, your answer may include the symbol \( I \), the moment of inertia of the assembly, whether or not you have answered the first part correctly.)

What is the angular acceleration \( \alpha \) of the rod immediately after it is released?

**Hint B.1 How to approach the problem**

The forces acting on the system (spheres and rod) are the weights of the spheres and the rod, and the reaction force from the pivot. Find the torque due to each of these forces about the pivot point and add them with the correct signs. Finally, use Newton's second law for rotational motion: \( \tau = I \alpha \).

**Hint B.2 Find the torque due to the sphere of mass \( m_1 \)**

Find the torque about the pivot due to the sphere of mass \( m_1 \).

**Hint B.2.1 Formula for torque**

The torque about the pivot point due to a force \( F \) is

\[
\tau = \mathbf{r}_{\text{pivot}} \times \mathbf{F} = r_{\text{pivot}} \sin \phi \mathbf{e}_z \mathbf{F}.
\]

where \( \mathbf{r}_{\text{pivot}} \) is the vector from the pivot point to the point where the force is applied. The other symbols have their usual meanings. If you are using any of the latter two expressions, you must remember that if the force tends to cause a clockwise rotation, you need to include a negative sign in your expression since the torque due to such a force is taken to be negative (by convention).

Express your answer in terms of given quantities. Keep in mind that the positive direction is counterclockwise.

**ANSWER:**

**Hint B.3 Find the torque due to the sphere of mass \( m_2 \)**

Find the torque about the pivot due to the particle of mass \( m_2 \).

Express your answer in terms of given quantities. Keep in mind that the positive direction is counterclockwise.

**ANSWER:**

**Hint B.4 Torque due to forces acting on the rod**

Besides the two masses, there are two more forces to consider: the normal force acting at the pivot and the gravitational force acting on the rod. The normal force acts at the pivot point, so its distance from the pivot point is zero, and thus this force contributes zero torque. The gravitational force acts at the rod's center of mass, which is also at the pivot point. Therefore, the torque due to the gravitational force about the pivot point is also zero for the rod.
Hint B.5 Relating the angular acceleration to the net torque

Let the net torque acting on the system about the pivot point be denoted by \( \tau_{\text{net}} \). Find an expression for \( \tau_{\text{net}} \).

Express your answer in terms of the system's moment of inertia \( I \) and its resulting angular acceleration \( \alpha \). (Use \( I \) in your answer, not the expression for \( I \) you found in Part A.)

**ANSWER:**

\[ \tau_{\text{net}} = \alpha I \]

Take the counterclockwise direction to be positive. Express \( \alpha \) in terms of some or all of the variables \( m_2 \), \( m_3 \), \( m_4 \), \( x_2 \), \( I \), and \( g \).

**ANSWER:**

\[ \alpha = \frac{(m_1 - m_2)g}{I} \frac{mx}{(\frac{m_1 - m_2}{m_1} + m_3 + m_4)} \]

Substituting for \( I \), the value obtained in Part A yields

\[ \alpha = \frac{(m_1 - m_2)g}{\frac{m_1}{m_1 - m_2} + m_3 + m_4} \]

A large angular acceleration is often desirable. This can be accomplished by making the connecting rod light and short (since both \( m_1 \) and \( x_2 \) appear in the denominator of the expression for \( \alpha \)). For a seesaw, on the other hand, \( m_1 \) and \( x_2 \) are usually chosen to be as large as possible, while making sure that the "rod" does not get too heavy and unwieldy. This ensures that the angular acceleration is quite low.

**Problem 11.51**

**Description:** An m crate is pulled up a 30 degree(s) incline by a rope angled \( \alpha \) above the incline. The tension in the rope is \( T \) and the crate's coefficient of kinetic friction on the incline is \( \mu \).

(a) How much work is done by tension, by gravity, and by...

An 7.7 kg crate is pulled 5.2 m up a 30° incline by a rope angled 19° above the incline. The tension in the rope is 140 N and the crate's coefficient of kinetic friction on the incline is 0.26.

**Part A**

How much work is done by tension, by gravity, and by the normal force?

Express your answers using two significant figures. Enter your answers numerically separated by commas.

**ANSWER:**

\[ W_T, W_g, W_n = m \cdot 9.8 \cdot 5.2 \cdot \cos(\frac{\pi}{3}) - \frac{\pi}{6}, J \]

**Part B**

What is the increase in thermal energy of the crate and incline?

Express your answer using two significant figures.

**ANSWER:**

\[ \Delta E_{\text{kin}} = \mu \left( m \cdot 9.8 \cdot \cos(\frac{\pi}{6}) - T \sin(\alpha) \right), J \]

**Where's the Energy?**

**Description:** A series of basic conceptual and computational questions about the motion of a block sliding down a ramp. Motion with and without the non-conservative forces is considered.

**Learning Goal:** To understand how to apply the law of conservation of energy to situations with and without nonconservative forces acting.

The law of conservation of energy states the following:

In an isolated system the total energy remains constant.
If the objects within the system interact through gravitational and elastic forces only, then the total mechanical energy is conserved.

The mechanical energy of a system is defined as the sum of kinetic energy $K$ and potential energy $U$. For such systems where no forces other than the gravitational and elastic forces do work, the law of conservation of energy can be written as

$$K_i + U_i = K_f + U_f$$

where the quantities with subscript ‘i’ refer to the “initial” moment and those with subscript ‘f’ refer to the final moment. A wise choice of initial and final moments, which is not always obvious, may significantly simplify the solution.

The kinetic energy of an object that has mass $m$ and velocity $v$ is given by

$$K = \frac{1}{2}mv^2.$$

Potential energy, instead, has many forms. The two forms that you will be dealing with most often in this chapter are the gravitational and elastic potential energy. Gravitational potential energy is the energy possessed by elevated objects. For small heights, it can be found as

$$U_g = mgh,$$

where $m$ is the mass of the object, $g$ is the acceleration due to gravity, and $h$ is the elevation of the object above the zero level. The zero level is the elevation at which the gravitational potential energy is assumed to be (you guessed it) zero. The choice of the zero level is dictated by convenience; typically (but not necessarily), it is selected to coincide with the lowest position of the object during the motion explored in the problem.

Elastic potential energy is associated with stretched or compressed elastic objects such as springs. For a spring with a force constant $k$, stretched or compressed a distance $x$, the associated elastic potential energy is

$$U_e = \frac{1}{2}kx^2.$$

When all three types of energy change, the law of conservation of energy for an object of mass $m$ can be written as

$$\frac{1}{2}mv_i^2 + mgh_i + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + mgh_f + \frac{1}{2}kx_f^2 + W_{noncons}.$$

The gravitational force and the elastic force are two examples of conservative forces. What if nonconservative forces, such as friction, also act within the system? In that case, the total mechanical energy would change. The law of conservation of energy is then written as

$$\frac{1}{2}mv_i^2 + mgh_i + \frac{1}{2}kx_i^2 + W_{noncons} = \frac{1}{2}mv_f^2 + mgh_f + \frac{1}{2}kx_f^2,$$

where $W_{noncons}$ represents the work done by the nonconservative forces acting on the object between the initial and the final moments. The work $W_{noncons}$ is usually negative; that is, the nonconservative forces tend to decrease, or dissipate, the mechanical energy of the system.

In this problem, we will consider the following situation as depicted in the diagram: A block of mass $m$ slides at a speed $v_i$ along a horizontal, smooth table. It then slides down a smooth ramp, descending a height $h$, and then slides along a horizontal rough floor, stopping eventually. Assume that the block slides slowly enough so that it does not lose contact with the supporting surfaces (table, ramp, or floor).

You will analyze the motion of the block at different moments using the law of conservation of energy.

![Diagram of block sliding](http://session.masteringphysics.com/myct/assignme...)

**Part A**

Which word in the statement of this problem allows you to assume that the table is frictionless?

**ANSWER:**

- straight
- smooth
- horizontal

Although there are no truly "frictionless" surfaces, sometimes friction is small enough to be neglected. The word "smooth" often describes such low-friction surfaces. Can you deduce what the word "rough" means?

**Part B**

Suppose the potential energy of the block at the table is given by $mgh/3$. This implies that the chosen zero level of potential energy is ________.
Hint B.1 Definition of $U$

Gravitational potential energy is given by

$$U_g = mgh,$$

where $h$ is the height relative to the zero level. Note that $h > 0$ when the object is above the chosen zero level; $h < 0$ when the object is below the chosen zero level.

**ANSWER:**

- a distance $h/3$ above the floor
- a distance $h/3$ below the floor
- a distance $2h/3$ above the floor
- a distance $2h/3$ below the floor
- on the floor

**Part C**

If the zero level is a distance $2h/3$ above the floor, what is the potential energy $U$ of the block on the floor?

Express your answer in terms of some or all the variables $m$, $v$, and $h$ and any appropriate constants.

**ANSWER:**

$$U = -\frac{2mgh}{3}$$

**Part D**

Considering that the potential energy of the block at the table is $mgh/3$ and that on the floor is $-2mgh/3$, what is the change in potential energy $\Delta U$ of the block if it is moved from the table to the floor?

**Hint D.1 Definition of $\Delta U$**

By definition, the change in potential energy is given by $\Delta U = U_f - U_i$. In general, change is always defined as the "final" quantity minus the "initial" one.

Express your answer in terms of some or all the variables $m$, $v$, and $h$ and any appropriate constants.

**ANSWER:**

$$\Delta U = -mgh$$

As you may have realized, this choice of the zero level was legitimate but not very convenient. Typically, in such problems, the zero level is assumed to be on the floor. In solving this problem, we will assume just that: the zero level of potential energy is on the floor.

**Part E**

Which form of the law of conservation of energy describes the motion of the block when it slides from the top of the table to the bottom of the ramp?

**Hint E.1 How to approach the problem**

Think about these questions:
- Are there any nonconservative forces acting on the block during this part of the trip?
- Are there any objects involved that can store elastic potential energy?
- Is the block changing its height?
- Is the block changing its speed?

**ANSWER:**

- $\frac{1}{2}mv^2 + mgh_i + W_{ext} = \frac{1}{2}mv^2 + mg(h_i + \frac{1}{2}x_f^2)$
- $\frac{1}{2}mv^2 + \frac{1}{2}kx_i^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx_f^2$
- $\frac{1}{2}mv^2 + mgh_i = mgh_i + \frac{1}{2}kx_f^2$
- $\frac{1}{2}mv^2 + mgh_i + \frac{1}{2}kx_i^2 = \frac{1}{2}mv^2 + mgh_i + \frac{1}{2}kx_f^2$
As the block slides down the ramp, what happens to its kinetic energy $K$, potential energy $U$, and total mechanical energy $E$?

**ANSWER:**
- $K$ decreases; $U$ increases; $E$ stays the same
- $K$ decreases; $U$ increases; $E$ increases
- $K$ increases; $U$ increases; $E$ increases
- $K$ increases; $U$ decreases; $E$ stays the same

**Part G**

Using conservation of energy, find the speed $v_b$ of the block at the bottom of the ramp.

**Hint G.1 How to approach the problem**
Use the equation for the law of conservation of energy that describes the motion of the block as it slides down the ramp. Then substitute in all known values and solve for the unknown.

Express your answer in terms of some or all the variables $m$, $v$, and $h$ and any appropriate constants.

**ANSWER:**
$$v_b = \sqrt{(2gh)}$$

**Part H**

Which form of the law of conservation of energy describes the motion of the block as it slides on the floor from the bottom of the ramp to the moment it stops?

**Hint H.1 How to approach the problem**
Think about these questions:
- Are there any nonconservative forces acting on the block during this part of the trip?
- Are there any objects involved that can store elastic potential energy?
- Is the block changing its height?
- Is the block changing its speed?

**ANSWER:**
- $\frac{1}{2}mv^2 + mgh \oplus W_{nc} = \frac{1}{2}mv_f^2 + mgh_f$
- $\frac{1}{2}mv^2 \oplus \frac{1}{2}mv_f^2$
- $\frac{1}{2}mv^2 + W_{nc} = \frac{1}{2}mv_f^2$
- $\frac{1}{2}mv^2 + mgh = \frac{1}{2}mv_f^2 + mgh_f$
- $\frac{1}{2}mv^2 + mg(h) + \frac{1}{2}kx^2 + W_{nc} = \frac{1}{2}mv_f^2 + mg(h_f) + \frac{1}{2}kx_f^2$

**Part I**

As the block slides across the floor, what happens to its kinetic energy $K$, potential energy $U$, and total mechanical energy $E$?

**ANSWER:**
- $K$ decreases; $U$ increases; $E$ decreases
- $K$ increases; $U$ increases; $E$ decreases
- $K$ decreases; $U$ stays the same; $E$ decreases
- $K$ increases; $U$ stays the same; $E$ decreases
- $K$ decreases; $U$ increases; $E$ stays the same
- $K$ increases; $U$ decreases; $E$ stays the same

**Part J**

What force is responsible for the decrease in the mechanical energy of the block?

**ANSWER:**
- tension
- gravity
- friction
- normal force
Part K
Find the amount of energy $E$ dissipated by friction by the time the block stops.

**Hint K.1 How to approach the problem**
Use the equation for the law of conservation of energy that you selected as the most appropriate for the block sliding on the floor. Then substitute in all known values and solve for the unknown. You will need to use the value for $v_f$ that you found earlier in Part G, as your initial speed.

**Express your answer in terms of some or all the variables $m$, $v_i$, and $f$ and any appropriate constants.**

**ANSWER:**
$$E = 0.5mv_i^2 + mgh$$

---

**Vector Dot Product**

**Description:** This problem reviews the rules for computing dot products and provides practice on calculating them.

Let vectors $\vec{A} = (2, 1, -4)$, $\vec{B} = (-3, 0, 1)$, and $\vec{C} = (-1, -1, 2)$.

Calculate the following:

**Part A**
**Hint A.1 Remember the dot product equation**
If $\vec{A} = (A_x, A_y, A_z)\text{ and }\vec{B} = (B_x, B_y, B_z)$ then

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

**ANSWER:**
$$\vec{A} \cdot \vec{B} = -10$$

**Part B**
What is the angle $\theta_{AB}$ between $\vec{A}$ and $\vec{B}$?

**Hint B.1 Remember the definition of dot products**

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos(\theta), \text{ where } \theta \text{ is the angle between } \vec{A} \text{ and } \vec{B}$$

**ANSWER:**
$$\theta_{AB} = 2.33 \text{ radians}$$

**Part C**

**ANSWER:**
$$2\vec{B} \cdot 3\vec{C} = 30$$

**Part D**

**ANSWER:**
$$2(\vec{B} \cdot 3\vec{C}) = 30$$

**Part E**
Which of the following can be computed?

**Hint E.1 Dot product operator**
The dot product operates only on two vectors. The dot product of a vector and a scalar is not defined.
Part F

Hint F.1 What is the angle between a vector and itself?
The angle between a vector and itself is 0.

Hint F.2 Remember the definition of dot products
\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos(\theta) \text{, where } \theta \text{ is the angle between } \vec{A} \text{ and } \vec{B}. \]

Express your answer in terms of \( V_i \)

\[
\vec{V}_i \cdot \vec{V}_i = \frac{V_i^2}{\bar{V}_i^2} \]

Part G

If \( \vec{V}_1 \) and \( \vec{V}_2 \) are perpendicular,

Hint G.1 What is the angle between perpendicular vectors?
The angle between vectors that are perpendicular is equal to \( \pi/2 \) radians or 90 degrees.

\[
\vec{V}_1 \cdot \vec{V}_2 = V_1 V_2 \cos \left( \frac{\pi}{2} \right) \]

Part H

If \( \vec{V}_1 \) and \( \vec{V}_2 \) are parallel,

Hint H.1 What is the angle between parallel vectors?
The angle between vectors that are parallel is equal to 0.

Express your answer in terms of \( V_i \) and \( V_x \)

\[
\vec{V}_1 \cdot \vec{V}_2 = V_1 V_2 \]

Problem 11.53

Description: A 48.0 kg ice skater is gliding along the ice, heading due north at 4.30 m/s. The ice has a small coefficient of static friction, to prevent the skater from slipping sideways, but \( \mu_k = 0 \). Suddenly, a wind from the northeast exerts a force of 4.30 N on the skater.

Part A

Use work and energy to find the skater's speed after gliding 100 m in this wind.
Part B

What is the minimum value of $\mu$ that allows her to continue moving straight north?

ANSWER: $\frac{F_{\text{com}}(\cos \theta)}{m_9 \mu}$

Center of Mass and External Forces

**Description:** Find the position, velocity, acceleration of the center of mass of a system of two blocks; compute the effect of internal and external forces on the acceleration of the center of mass.

**Learning Goal:** Understand that, for many purposes, a system can be treated as a point-like particle with its mass concentrated at the center of mass.

A complex system of objects, both point-like and extended ones, can often be treated as a point particle, located at the system’s center of mass. Such an approach can greatly simplify problem solving.

Before you use the center of mass approach, you should first understand the following terms:

- **System:** Any collection of objects that are of interest to you in a particular situation. In many problems, you have a certain freedom in choosing your system. Making a wise choice for the system is often the first step in solving the problem efficiently.

- **Center of mass:** The point that represents the “average” position of the entire mass of a system. The position of the center of mass $\vec{r}_{\text{com}}$ can be expressed in terms of the position vectors $\vec{r}_i$ of the particles as

$$\vec{r}_{\text{com}} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

The $x$ coordinate of the center of mass $x_{\text{com}}$ can be expressed in terms of the $x$ coordinates $(r_x)$ of the particles as

$$x_{\text{com}} = \frac{\sum m_i (x_i)}{\sum m_i}$$

Similarly, the $y$ coordinate of the center of mass can be expressed.

- **Internal force:** Any force that results from an interaction between the objects inside your system. As we will show, the internal forces do not affect the motion of the system’s center of mass.

- **External force:** Any force acting on an object inside your system that results from an interaction with an object outside your system.

Consider a system of two blocks that have masses $m_1$ and $m_2$. Assume that the blocks are point-like particles and are located along the $x$ axis at the coordinates $x_1$ and $x_2$ as shown.

In this problem, the blocks can only move along the $x$ axis.

**Part A**

Find the $x$ coordinate $x_{\text{com}}$ of the center of mass of the system.

Express your answer in terms of $m_1$, $m_2$, $x_1$, and $x_2$.

**ANSWER:**

$$x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$
Part B
If \( m_2 \gg m_1 \), then the center of mass is located:

ANSWER:
- to the left of \( m_1 \) at a distance much greater than \( x_2 - x_1 \)
- to the left of \( m_1 \) at a distance much less than \( x_2 - x_1 \)
- to the right of \( m_1 \) at a distance much less than \( x_2 - x_1 \)
- to the right of \( m_1 \) at a distance much greater than \( x_2 - x_1 \)
- to the left of \( m_2 \) at a distance much less than \( x_2 - x_1 \)
- to the right of \( m_2 \) at a distance much less than \( x_2 - x_1 \)

Part C
If \( m_2 = m_1 \), then the center of mass is located:

ANSWER:
- at \( m_1 \)
- at \( m_2 \)
- half-way between \( m_1 \) and \( m_2 \)
- the answer depends on \( x_1 \) and \( x_2 \)

Part D
Recall that the blocks can only move along the x axis. The x components of their velocities at a certain moment are \( v_{1x} \) and \( v_{2x} \). Find the x component of the velocity of the center of mass \( (v_{cm})_x \) at that moment. Keep in mind that, in general: \( v_{cm} = \frac{dx}{dt} \)

Express your answer in terms of \( m_1 \), \( v_{1x} \), \( m_2 \), and \( v_{2x} \)

ANSWER:
\[
(v_{cm})_x = \frac{m_1 v_{1x} + m_2 v_{2x}}{m_1 + m_2}
\]

Because \( v_{1x} \) and \( v_{2x} \) are the x components of the velocities of \( m_1 \) and \( m_2 \), their values can be positive or negative or equal to zero.

Part E
Suppose that \( v_{1x} \) and \( v_{2x} \) have equal magnitudes. Also, \( v_{1x}^2 \) is directed to the right and \( v_{2x}^2 \) is directed to the left. The velocity of the center of mass is then:

ANSWER:
- directed to the left
- directed to the right
- zero
- the answer depends on the ratio \( \frac{m_2}{m_1} \)

Part F
Assume that the x components of the blocks’ momenta at a certain moment are \( p_{1x} \) and \( p_{2x} \). Find the x component of the velocity of the center of mass \( (v_{cm})_x \) at that moment.

Express your answer in terms of \( m_1 \), \( m_2 \), \( p_{1x} \), and \( p_{2x} \)

ANSWER:
\[
(v_{cm})_x = \frac{p_{1x} + p_{2x}}{m_1 + m_2}
\]

Part G
Suppose that \( \vec{p}_{cm} = 0 \). Which of the following must be true?

ANSWER:
- \( |p_{1x}| = |p_{2x}| \)
- \( |p_{1x}| = |p_{2x}| \)
- \( m_1 = m_2 \)
- none of the above

Part H
Assume that the blocks are accelerating, and the $x$ components of their accelerations at a certain moment are $a_{1x}$ and $a_{2x}$. Find the $x$ component of the acceleration of the center of mass $(a_{cm})_x$ at that moment. Keep in mind that, in general, $a = \frac{dv}{dt}$.

Express your answer in terms of $m_1$, $m_2$, $a_{1x}$, and $a_{2x}$.

**ANSWER:**

$$(a_{cm})_x = \frac{m_1a_{1x} + m_2a_{2x}}{m_1 + m_2}$$

Because $a_{1x}$ and $a_{2x}$ are the $x$ components of the velocities of $m_1$ and $m_2$, their values can be positive or negative or equal to zero.

We will now consider the effect of external and internal forces on the acceleration of the center of mass.

**Part I**

Consider the same system of two blocks. An external force $\vec{F}$ is now acting on block $m_1$. No forces are applied to block $m_2$ as shown. Find the $x$ component of the acceleration of the center of mass $(a_{cm})_x$ of the system.

**Hint I.1 Using Newton’s laws**

Find the acceleration of each block from Newton’s second law and then apply the formula for $(a_{cm})_x$ found earlier.

Express your answer in terms of the $x$ component of the force, $F_x$, and $m_1$, $m_2$.

**ANSWER:**

$$(a_{cm})_x = \frac{F_x}{m_1 + m_2}$$

**Part J**

Consider the same system of two blocks. Now, there are two forces involved. An external force $\vec{F}_1$ is acting on block $m_1$ and another external force $\vec{F}_2$ is acting on block $m_2$. Find the $x$ component of the acceleration of the center of mass $(a_{cm})_x$ of the system.

Express your answer in terms of the $x$ components of the forces, $F_{1x}$ and $F_{2x}$, and $m_1$, $m_2$.

**ANSWER:**

$$(a_{cm})_x = \frac{F_{1x} + F_{2x}}{m_1 + m_2}$$

Note that, in both cases, the acceleration of mass can be found as

$$(a_{cm})_x = \frac{(\sum F_{ext})_x}{M_{total}}$$

where $F_{ext}$ is the net external force applied to the system, and $M_{total}$ is the total mass of the system. Even though each force is only applied to one object, it affects the acceleration of the center of mass of the entire system.
This result is especially useful since it can be applied to a general case, involving any number of objects moving in all directions and being acted upon by any number of external forces.

Part K
Consider the previous situation. Under what condition would the acceleration of the center of mass be zero? Keep in mind that \( F_{x1} \) and \( F_{x2} \) represent the components, of the corresponding forces.

**ANSWER:**
- \( F_{x1} = -F_{x2} \)
- \( F_{x1} = F_{x2} \)
- \( m_1 = m_2 \)
- \( m_1 \neq m_2 \)

Part L
Consider the same system of two blocks. Now, there are two internal forces involved. An internal force \( \vec{F}_{12} \) is applied to block \( m_1 \) by block \( m_2 \) and another internal force \( \vec{F}_{21} \) is applied to block \( m_2 \) by block \( m_1 \). Find the \( x \) component of the acceleration of the center of mass \( \{a_{cm}\}_x \) of the system.

Express your answer in terms of the \( x \) components \( F_{12x} \) and \( F_{21x} \) of the forces, \( m_1 \) and \( m_2 \).

**ANSWER:**
\[
\frac{F_{12x} + F_{21x}}{m_1 + m_2} = 0
\]

Newton’s 3rd law tells you that \( |F_{12x}| = -|F_{21x}| \). From your answers above, you can conclude that \( \{a_{cm}\}_x = 0 \). The internal forces do not change the velocity of the center of mass of the system. In the absence of any external forces, \( \{a_{cm}\} = 0 \) and \( \{a_{cm}\}_x \) is constant.

You just demonstrated this to be the case for the two-body situation moving along the \( x \) axis; however, it is true in more general cases as well.

**Introduction to Moments of Inertia**

**Description:** Conceptual questions about moment of inertia; several basic computational questions for both discrete and continuous mass distribution.

**Learning Goal:** To understand the definition and the meaning of moment of inertia; to be able to calculate the moments of inertia for a group of particles and for a continuous mass distribution with a high degree of symmetry.

By now, you may be familiar with a set of equations describing rotational kinematics. One thing that you may have noticed was the similarity between translational and rotational formulas. Such similarity also exists in dynamics and in the work-energy domain.

For a particle of mass \( m \) moving at a constant speed \( v \), the kinetic energy is given by the formula \( K = \frac{1}{2}mv^2 \). If we consider instead a rigid object of mass \( M \) rotating at a constant angular speed \( \omega \), the kinetic energy of such an object cannot be found by using the formula \( K = \frac{1}{2}Mv^2 \) directly: different parts of the object have different linear speeds. However, they all have the same angular speed. It would be desirable to obtain a formula for kinetic energy of rotational motion that is similar to the one for translational motion; such a formula would include the term \( \frac{1}{2}I\omega^2 \) instead of \( \frac{1}{2}Mv^2 \).

Such a formula can, indeed, be written: for rotational motion of a system of small particles or for a rigid object with continuous mass distribution, the kinetic energy can be written as

\[
K = \frac{1}{2}I\omega^2
\]

Here, \( I \) is called the moment of inertia of the object (or of the system of particles). It is the quantity representing the inertia with respect to rotational motion.
It can be shown that for a discrete system, say of \( n \) particles, the moment of inertia (also known as rotational inertia) is given by

\[
I = \sum_{i=1}^{n} m_i r_i^2
\]

In this formula, \( m_i \) is the mass of the \( i \)th particle and \( r_i \) is the distance of that particle from the axis of rotation.

For a rigid object, consisting of infinitely many particles, the analogue of such summation is integration over the entire object:

\[
I = \int r^2 \, dm
\]

In this problem, you will answer several questions that will help you better understand the moment of inertia, its properties, and its applicability. It is recommended that you read the corresponding sections in your textbook before attempting these questions.

**Part A**

On which of the following does the moment of inertia of an object depend?

Check all that apply.

**ANSWER:**

- linear speed
- linear acceleration
- angular speed
- angular acceleration
- total mass
- shape and density of the object
- location of the axis of rotation

Unlike mass, the moment of inertia depends not only on the amount of matter in an object but also on the distribution of mass in space. The moment of inertia is also dependent on the axis of rotation. The same object, rotating with the same angular speed, may have different kinetic energy depending on the axis of rotation.

Consider the system of two particles, a and b, shown in the figure. Particle a has mass \( m_a \), and particle b has mass \( 2m_b \).

**Part B**

What is the moment of inertia of particle a?

**ANSWER:**

- undefined: an axis of rotation has not been specified.

**Part C**

Find the moment of inertia \( I_x \) of particle a with respect to the x axis (that is, if the x axis is the axis of rotation), the moment of inertia \( I_y \) of particle a with respect to the y axis, and the moment of inertia \( I_z \) of particle a with respect to the z axis (the axis that passes through the origin perpendicular to both the x and y axes).

Express your answers in terms of \( m_a \) and \( r \) separated by commas.

**ANSWER:**

\[
I_x, I_y, I_z = m_a r^2, 9m_a r^2, 10m_a r^2
\]
**Part D**

Find the total moment of inertia \( I \) of the system of two particles shown in the diagram with respect to the \( y \) axis.

Express your answer in terms of \( m_1 \) and \( r \).

**ANSWER:**

\[
I = \frac{11m_1r^2}{2}
\]

**Part E**

For parts E through G, suppose that both particles rotate with the same angular speed \( \omega \) about the \( y \) axis while maintaining their distances from the \( y \) axis.

**Part E**

Using the total moment of inertia \( I \) of the system found in Part D, find the total kinetic energy \( K \) of the system.

Express your answer in terms of \( m_1 \), \( \omega \), and \( r \).

**ANSWER:**

\[
K = \frac{11m_1\omega^2}{2}
\]

It is useful to see how the formula for rotational kinetic energy agrees with the formula \( K = \frac{1}{2}mr^2 \) for the kinetic energy of an object that is not rotating. To see the connection, let us find the kinetic energy of each particle.

**Part F**

Using the formula for kinetic energy of a moving particle \( K = \frac{1}{2}mr^2 \), find the kinetic energy \( K_a \) of particle a and the kinetic energy \( K_b \) of particle b.

**Hint F.1**

Find the linear speed

Using the formula \( v = \omega r \), find the linear speed \( v_a \) of particle a.

Express your answer in terms of \( \omega \) and \( r \).

**ANSWER:**

\[
v_a = \omega r
\]

Express your answers in terms of \( m_1 \), \( \omega \), and \( r \) separated by a comma.

**ANSWER:**

\[
K_a, K_b = \frac{9m_1(\omega r)^2}{2}, \frac{2m_1(\omega r)^2}{2}
\]

**Part G**

Using the results for the kinetic energy of each particle, find the total kinetic energy \( K \) of the system of particles.

Express your answer in terms of \( m_1 \), \( \omega \), and \( r \).

**ANSWER:**

\[
K = \frac{11m_1(\omega r)^2}{2}
\]

Not surprisingly, the formulas \( K = \frac{1}{2}I\omega^2 \) and \( K = \frac{1}{2}mr^2 \) give the same result. They should, of course, since the rotational kinetic energy of a system of particles is simply the sum of the kinetic energies of the individual particles making up the system.

**Constrained Rotation and Translation**

Description: Determine the constraint equations that relate linear to rotational motion for the cases of an unrolling tape measure and a rolling tire.
Learning Goal: To understand that contact between rolling objects and what they roll against imposes constraints on the change in position (velocity) and angle (angular velocity).

The way in which a body makes contact with the world often imposes a constraint relationship between its possible rotation and translational motion. A ball rolling on a road, a yo-yo unwinding as it falls, and a baseball leaving the pitcher's hand are all examples of constrained rotation and translation. In a similar manner, the rotation of one body and the translation of another may be constrained, as happens when a fireman unrolls a hose from its storage drum.

Situations like these can be modeled by constraint equations, relating the coupled angular and linear motions. Although these equations fundamentally involve position (the angle of the wheel at a particular distance down the road), it is usually the relationship of velocities and accelerations that are relevant in solving a problem involving such constraints. The velocities are needed in the conservation equations for momentum and angular momentum, and the accelerations are needed for the dynamical equations.

It is important to use the standard sign conventions: positive for counterclockwise rotation and positive for motion toward the right. Otherwise, your dynamical equations will have to be modified. Unfortunately, a frequent result will be the appearance of negative signs in the constraint equations.

Consider a measuring tape unwinding from a drum of radius $r$. The center of the drum is not moving; the tape unwinds as its free end is pulled away from the drum. Neglect the thickness of the tape, so that the radius of the drum can be assumed not to change as the tape unwinds. In this case, the standard conventions for the angular velocity $\omega(t)$ and for the (translational) velocity $v(t)$ of the end of the tape result in a constraint equation with a positive sign (e.g., if $\omega > 0$, that is, the tape is unwinding, then $\omega > 0$ also).

Part A
Assume that the function $x(t)$ represents the length of tape that has unwound as a function of time. Find $\theta(t)$, the angle through which the drum will have rotated, as a function of time.

**Hint A.1** Find the amount of tape that unrolls in one complete revolution of the drum
If the measuring tape unwinds one complete revolution ($\theta = 2\pi r$), how much tape, $x_{2\pi r}$, will have unwound?

**ANSWER:**

$x_{2\pi r} = 2\pi r$

Express your answer (in radians) in terms of $x(t)$ and any other given quantities.

**ANSWER:**

$\theta(t) = \frac{x(t)}{r}$ radians

Part B
The tape is now wound back into the drum at angular rate $\omega(t)$. With what speed will the end of the tape move? (Note that our drawing specifies that a positive derivative of $x(t)$ implies motion away from the drum. Be careful with your signs! The fact that the tape is being wound back into the drum implies that $\omega(t) < 0$, and for the end of the tape to move closer to the drum, it must be the case that $v(t) < 0$.

**Hint B.1** How to approach the problem
The function $\omega(t)$ is given by the derivative of $\theta(t)$ with respect to time. Compute this derivative using the expression for $\theta(t)$ found in Part A and the fact that $\frac{d\theta}{dt} = \frac{v(t)}{r}$.

Express your answer in terms of $v(t)$ and $r$.

**ANSWER:**

$\omega(t) = \frac{v(t)}{r}$

Answer in terms of $\omega(t)$ and other given quantities from the problem introduction.

**ANSWER:**

$v(t) = \omega(t) r$

Part C
Since $r$ is a positive quantity, the answer you just obtained implies that $v(t)$ will always have the same sign as $\omega(t)$ if the tape is unwinding, both quantities will be positive. If the
tape is being wound back up, both quantities will be negative. Now find \( \alpha(t) \), the linear acceleration of the end of the tape.

Express your answer in terms of \( \alpha(t) \), the angular acceleration of the drum: \( \alpha(t) = \frac{d\omega(t)}{dt} \).

**ANSWER:**
\[
\alpha(t) = \alpha(t) r
\]

---

**Part D**

Perhaps the trickiest aspect of working with constraint equations for rotational motion is determining the correct sign for the kinematic quantities. Consider a tire of radius \( r \) rolling to the right, without slipping, with constant \( x \) velocity \( \eta_x \). Find \( \omega \), the (constant) angular velocity of the tire. Be careful of the signs in your answer; recall that positive angular velocity corresponds to rotation in the counterclockwise direction.

Express your answer in terms of \( \eta_x \) and \( r \).

**ANSWER:**
\[
\omega = \frac{-\eta_x}{r}
\]

This is an example of the appearance of negative signs in constraint equations—a tire rolling in the positive direction translationally exhibits negative angular velocity, since rotation is clockwise.

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**Part E**

Assume now that the angular velocity of the tire, which continues to roll without slipping, is not constant, but rather that the tire accelerates with constant angular acceleration \( \alpha \). Find \( \alpha_e \), the linear acceleration of the tire.

Express your answer in terms of \( \omega \) and \( r \).

**ANSWER:**
\[
\alpha_e = -\alpha r
\]

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**Spinning the Wheels: An Introduction to Angular Momentum**

**Description:** A mix of qualitative and computational questions introducing angular momentum and its basic applications.

**Learning Goal:** To learn the definition and applications of angular momentum including its relationship to torque.

By now, you should be familiar with the concept of momentum, defined as the product of an object’s mass and its velocity:

\[
\vec{p} = m\vec{v}.
\]

You may have noticed that nearly every translational concept or equation seems to have an analogous rotational one. So, what might be the rotational analogue of momentum?

Just as the rotational analogue of force \( \vec{F} \), called the torque \( \vec{\tau} \), is defined by the formula

\[
\vec{\tau} = \vec{r} \times \vec{F},
\]

the rotational analogue of momentum \( \vec{p} \), called the angular momentum \( \vec{L} \), is given by the formula

\[
\vec{L} = \vec{r} \times \vec{p},
\]

for a single particle. For an extended body you must add up the angular momenta of all of the pieces.

There is another formula for angular momentum that makes the analogy to momentum particularly clear. For a rigid body rotating about an axis of symmetry, which will be true for all parts in this problem, the measure of inertia is given not by the mass \( m \) but by the rotational inertia (i.e., the moment of inertia) \( I \). Similarly, the rate of rotation is given by the body’s angular speed, \( \omega \). The product \( I \omega \) gives the angular momentum \( \vec{L} \) of a rigid body rotating about an axis of symmetry. (Note that if the body is not rotating about an axis of symmetry, then the angular momentum and the angular velocity may not be parallel.)
Part A
Which of the following is the SI unit of angular momentum?

ANSWER:
- N·m/s
- kg·m/s
- kg·m²/s²
- kg·m²/θ

Part B
An object has rotational inertia. The object, initially at rest, begins to rotate with a constant angular acceleration of magnitude \( \alpha \). What is the magnitude of the angular momentum \( L \) of the object after time \( t \)?

Hint B.1 How to approach the problem
Find the angular velocity first; then use the \( \vec{L} = \vec{r} \times \vec{p} \) definition of angular momentum.

Express your answer in terms of \( I \), \( \omega \), and \( t \).

ANSWER: \( L = I \alpha t \)

Part C
A rigid, uniform bar with mass \( m \) and length \( l \) rotates about the axis passing through the midpoint of the bar perpendicular to the bar. The linear speed of the end points of the bar is \( v \). What is the magnitude of the angular momentum \( L \) of the bar?

Hint C.1 How to approach the problem
Find separately the rotational inertia and the angular velocity; then use the \( \vec{L} = \vec{r} \times \vec{p} \) definition of angular momentum.

Hint C.2 Rotational inertia of the bar
For this axis, the rotational inertia of the bar is given by

\[ I = \frac{ml^2}{12} \]

where \( l \) is the length of the bar.

Hint C.3 Finding the magnitude of the angular velocity
The magnitude of the angular velocity \( \omega \) can be obtained from the relation

\[ \omega r = v \]

where \( r \) is the radius of rotation.

Express your answer in terms of \( m \), \( l \), \( v \), and appropriate constants.

ANSWER: \( L = \frac{mlv}{6} \)
You may recall that, according to Newton’s 2nd law, the rate of change of momentum of an object equals the net force acting on the object:
\[
\frac{d\mathbf{p}}{dt} = \mathbf{F}_{\text{net}}
\]

Similarly, the rate of change of angular momentum of an object equals the net torque acting on the object:
\[
\frac{d\mathbf{L}}{dt} = \mathbf{T}_{\text{net}}
\]

Therefore, if the net torque acting on an object (or a system of objects) is zero (i.e., the system is “closed”), then the rate of change of angular momentum is also zero. In other words, the net angular momentum of a closed system is constant (conserved).

This statement is known as the law of conservation of angular momentum. Just like the laws of conservation of energy and momentum, the law of conservation of angular momentum plays a major role in mechanics.

**Part D**

The uniform bar shown in the diagram has a length of 0.80 m. The bar begins to rotate from rest in the horizontal plane about the axis passing through its left end. What will be the magnitude of the angular momentum \( L \) of the bar 6.0 s after the motion has begun? The forces acting on the bar are shown.

**Hint D.1 How to approach the problem**

First, find the net torque, and then use

\[
\frac{d\mathbf{L}}{dt} = \mathbf{T}_{\text{net}}
\]

Express your answer in kg \( \cdot \) m\(^2\)/s to two significant figures.

**ANSWER:**

\[
L = 4.8 \text{ kg} \cdot \text{m}^2/\text{s}
\]

**Part E**

Each of the four bars shown can rotate freely in the horizontal plane about its left end. For which diagrams is the net torque equal to zero?

Type in alphabetical order the letters corresponding to the correct diagrams. For instance, if you think that only diagrams A, B, and C answer the question, type ABC.

**ANSWER:** BC
Part F

Consider the figures for Part E. For which diagrams is the angular momentum constant?

**Hint F.1 Determining when angular momentum is constant**

Recall that

\[
\frac{dL}{dt} = \tau_{net}
\]

This means that if the net torque is zero, then the angular momentum is a constant. That is, if the rate of change for \( L \) is zero, then \( L \) can't be changing.

Type alphabetically the letters corresponding to the correct diagrams. For instance, if you think that only diagrams A, B, and C answer the question, type ABC.

ANSWER: BC

Angular momentum is conserved when the net torque is zero. This is analogous to the statement from linear dynamics that momentum is conserved when the net force is zero.

Part G

Each of the disks in the figure has radius \( r \). Each disk can rotate freely about the axis passing through the center of the disk perpendicular to the plane of the figure, as shown. For which diagrams is the angular momentum constant? In your calculations, use the information provided in the diagrams.

Type alphabetically the letters corresponding to the correct diagrams. For instance, if you think that only diagrams A, B, and C answer the question, type ABC.

ANSWER: AD

Part H

Three disks are spinning independently on the same axle without friction. Their respective rotational inertias and angular speeds are \( I, \omega_1 \) (clockwise); \( 2I, \omega_2 \) (counterclockwise); and \( 4I, \omega_3/2 \) (clockwise). The disks then slide together and stick together, forming one piece with a single angular velocity. What will be the direction and the rate of rotation \( \omega_{\text{final}} \) of the single piece?

**Hint H.1 How to approach the problem**

The angular momentum for the system of the three disks is conserved. Therefore, if you find the angular momentum of the system before the disks stick together and the moment of inertia after they have stuck, you can solve for the angular speed \( \omega_{\text{final}} \) of the single piece.

**Hint H.2 Find the rotational inertia**

What is the rotational inertia \( I_{\text{net}} \) of the single piece?

Express your answer in terms of \( I \).

ANSWER: \( I_{\text{net}} = 7I \)

Express your answer in terms of one or both of the variables \( I \) and \( \omega \) and appropriate constants. Use a minus sign for clockwise rotation.

ANSWER: \( \omega_{\text{final}} = \frac{8\omega}{7} \)
Torque about the z Axis

Description: Guides student through both the “tangential force component” and “moment arm” methods for finding torque.

Learning Goal: To understand two different techniques for computing the torque on an object due to an applied force.

Imagine an object with a pivot point p at the origin of the coordinate system shown. The force vector $\vec{F}$ lies in the $xy$ plane, and this force of magnitude $F$ acts on the object at a point in the $xy$ plane. The vector $\vec{r}$ is the position vector relative to the pivot point p to the point where $\vec{F}$ is applied.

The torque on the object due to the force $\vec{F}$ is equal to the cross product $\tau = \vec{r} \times \vec{F}$. When, as in this problem, the force vector and lever arm both lie in the $xy$ plane of the paper or computer screen, only the $z$ component of torque is nonzero.

When the torque vector is parallel to the $z$ axis ($\vec{\tau} = \hat{z}$), it is easiest to find the magnitude and sign of the torque, $\tau$, in terms of the angle $\theta$ between the position and force vectors using one of two simple methods: the Tangential Component of the Force method or the Moment Arm of the Force method.

Note that in this problem, the positive $z$ direction is perpendicular to the computer screen and points toward you (given by the right-hand rule), so a positive torque would cause counterclockwise rotation about the $z$ axis.

Tangential component of the force

Part A

Decompose the force vector $\vec{F}$ into radial (i.e., parallel to $\vec{r}$) and tangential (perpendicular to $\vec{r}$) components as shown.

Find the magnitude of the radial and tangential components, $F_r$ and $F_t$. You may assume that $\theta$ is between zero and 90 degrees.

Hint A.1 Magnitude of $F_r$

Use the given angle between the force vector $\vec{F}$ and its radial component $\vec{F}_r$ to compute the magnitude $F_r$.

Enter your answer as an ordered pair. Express $F_r$ and $F_t$ in terms of $F$ and $\theta$.

**ANSWER:**

\[
(F_r, F_t) = \begin{cases} 
F \cos(\theta) \\
F \sin(\theta) 
\end{cases}
\]

Part B

Is the following statement true or false?

The torque about point p is proportional to the length $r$ of the position vector $\vec{r}$.

**ANSWER:**

true  false

Part C

Is the following statement true or false?

Both the radial and tangential components of $\vec{F}$ generate torque about point p.

**ANSWER:**

true  false
Part D
Is the following statement true or false?
In this problem, the tangential force vector would tend to turn an object clockwise around pivot point p.

ANSWER: [ ] true  [ ] false

Part E
Find the torque \( \tau \) about the pivot point p due to force \( \vec{F} \). Your answer should correctly express both the magnitude and sign of \( \tau \).

Express your answer in terms of \( F_i \) and \( \theta \) or in terms of \( \vec{r}, \vec{\theta} \), and \( \vec{F} \).

ANSWER: 
\[ \tau = -\tau F_i \]

Moment arm of the force
In the figure, the dashed line extending from the force vector is called the line of action of \( \vec{F} \). The perpendicular distance \( r_m \) from the pivot point p to the line of action is called the moment arm of the force.

Part F
What is the length, \( r_m \), of the moment arm of the force \( \vec{F} \) about point p?

Express your answer in terms of \( F_i \) and \( \theta \).

ANSWER: 
\[ r_m = r \sin(\theta) \]

Part G
Find the torque \( \tau \) about p due to \( \vec{F} \). Your answer should correctly express both the magnitude and sign of \( \tau \).

Express your answer in terms of \( r_m \) and \( F \) or in terms of \( \vec{r}, \vec{\theta} \), and \( \vec{F} \).

ANSWER: 
\[ \tau = -\tau r_m F \]

Three equivalent expressions for expressing torque about the \( z \) axis have been discussed in this problem:

1. Torque is defined as the cross product between the position and force vectors. When both \( \vec{r} \) and \( \vec{F} \) lie in the \( xy \) plane, only the \( z \) component of torque is nonzero, and the cross product simplifies to:
   \[ \vec{\tau} = \vec{r} \times \vec{F} = \tau \hat{z} \]
   Note that a positive value for \( \tau \) indicates a counterclockwise direction about the \( z \) axis.

2. Torque is generated by the component of \( \vec{F} \) that is tangential to the position vector \( \vec{r} \) (the tangential component of force):
3. The magnitude of torque is the product of the force and the perpendicular distance between the \( z \) axis and the line of action of a force, \( r_{\text{arm}} \), called the moment arm of the force:

\[
\tau = r_{\text{arm}} \times F = r \times F \sin(\theta).
\]