Angular Momentum - 2

Physics 201, Fall 2009
Chapter 10 – Lecture 2

- Angular momentum and torque about points (instead of axis)
- Gyroscope revisited
- Angular momentum of freely moving particles
- More angular momentum examples

Angular momentum about a point

The angular momentum vector about a point does not necessarily point parallel to the axis of rotation, only the z-component (if defined along the axis).

\[ \vec{L}_z = I \omega \]

Where \( I \) is computed with respect to axis of rotation (but it is not the axis of symmetry)

It does if the rotation is about axis of symmetry:

\[ \vec{L} = I \vec{\omega} \]
The angular momentum of a body about a particular axis as a function of time is shown in the graph. The external torque acting on the body along this axis at $t = 2$ s is

A. 0 
B. 5 N · m 
C. 10 N · m
Particles 1, 2, and 3 have equal masses and equal speeds. The angular momentum with respect to the origin for these three masses is

A. greatest for particle 1.
B. greatest for particle 2.
C. greatest for particle 3.
D. least for particle 2.
Torque about a point (origin)

Torque (always):
\[ \vec{\tau} = \vec{r} \times \vec{F} \]

Torque may not be parallel to z-axis.
The z-component of the torque
\[ \tau_z = r_{rad} \times F_{xy} \]

The corresponding angular momentum equations:
\[ \vec{L} = \vec{r} \times \vec{p} \]
\[ \vec{L}_z = \vec{r}_{rad} \times \vec{p}_{xy} \]

Spin and orbital angular momentum

\[ \vec{L}_{Sys} = \vec{L}_{Orbit} + \vec{L}_{Spin} \]

Total angular momentum = vector sum of spin and orbital momentum
\[ \vec{L}_{Orbit} = \vec{r}_{cm} \times M \vec{v} \]
The Gyroscope
\[ \mathbf{\tau} = \mathbf{r}_{cm} \times \mathbf{g} \]

Let's calculate the precession frequency

\[ \Delta \theta_{\text{precession}} = \frac{\Delta L}{L} \]

L forms a circular motion: \( \Delta L = L \Delta \theta_{\text{precession}} \)

The precession frequency

\[ \omega_{\text{precession}} = \frac{\Delta \theta_{\text{precession}}}{\Delta t} = \frac{\Delta L}{L \Delta t} = \frac{\tau_s}{L} = \frac{r_{cm} Mg}{I \omega_{\text{Wheel}}} \]
**Question:**
We repeat the gyroscope experiment on the moon ($g_{\text{moon}} = \frac{1}{6} g_{\text{Earth}}$) but with an angular velocity double from the one in the lecture.
Will the precession of the wheel be faster, slower or the same?

\[ \tau = r_{\text{cm}} \times g \]
\[ \tau = \frac{dL}{dt} \]

\[ \omega_{\text{precession}} = \frac{\Delta \theta_{\text{precession}}}{\Delta t} = \frac{\Delta L}{L \Delta t} = \frac{r_{\text{cm}} g}{I \omega_{\text{Wheel}}} = \frac{r_{\text{cm}} M}{I} \frac{g}{\omega_{\text{Wheel}}} \]

\[ \frac{\omega_{\text{Wheel, moon}}}{\omega_{\text{Wheel, Earth}}} = 1 \frac{1}{6} \frac{\omega_{\text{Wheel, lecture}}}{\omega_{\text{Wheel, lecture}}} = \frac{1}{3} \frac{d\phi}{d\tau} \]

**Angular Momentum of a Freely Moving Particle**

- We defined the angular momentum of a particle about the origin as
  \[ \vec{L} = \vec{r} \times \vec{p} \]

- This expression does **not** demand that the particle is moving in a circle.

- It’s angular momentum is still conserved.
Angular Momentum of a Freely Moving Particle

Consider a particle of mass $m$ moving with speed $v$ along the line $y = -d$. What is its angular momentum as measured from the origin $(0,0)$?

Angular Momentum of a Freely Moving Particle

- We need to figure out $\vec{L} = \vec{r} \times \vec{p}$
- The magnitude of the angular momentum is:
  $$|\vec{L}| = |\vec{r} \times \vec{p}| = rp \sin \theta = p[r \sin \theta] = pd = p \times (\text{distance of closest approach})$$
- Since $\vec{r}$ and $\vec{p}$ are both in the $x$-$y$ plane, $\vec{L}$ will be in the $z$ direction (right hand rule):
  $$L_z = pd$$
Angular Momentum of a Freely Moving Particle

- So we see that the direction of $\mathbf{L}$ is along the $z$ axis, and its magnitude is given by $L_z = pd = mvd$.
- $L$ is clearly conserved since $d$ is constant (the distance of closest approach of the particle to the origin) and $p$ is constant (momentum conservation).

Example: bullet hitting a stick

- A uniform stick of mass $m$ and length $D$ is pivoted at the center. A bullet of mass $m$ is shot through the stick at a point halfway between the pivot and the end. The initial speed of the bullet is $v_1$, and the final speed is $v_2$.

- What is the angular velocity $\omega_f$ of the stick after the collision?
Example: bullet hitting a stick

Initial angular momentum:
\[ L_i = px = mv_1 \frac{D}{4} \]

Final angular momentum:
\[ L_f = mv_2 \frac{D}{4} + I\omega_f = mv_2 \frac{D}{4} + \frac{1}{12} MD^2 \omega_f \]

Conservation of angular momentum around pivot axis:
\[ L_i = L_f \]
\[ mv_1 = mv_2 + \frac{1}{3} MD\omega_f \]
\[ \omega_f = \frac{3m}{M} D(v_1 - v_2) \]

Example: throw ball from stool

- A student sits on a stool which is free to rotate. The moment of inertia of the student plus the stool is \( I \). She throws a heavy ball of mass \( M \) with speed \( v \) such that its velocity vector passes a distance \( d \) from the axis of rotation.
- What is the angular speed \( \omega_f \) of the student-stool system after she throws the ball?

\[ L_i = 0 = L_f \]
\[ L_f = I\omega_f + Mvd \]
\[ \omega_f = \frac{Mvd}{I} \]
Angular momentum

A student is riding on the outside edge of a merry-go-round rotating about a frictionless pivot. She holds a heavy ball at rest in her hand. If she releases the ball, the angular velocity of the merry-go-round will:

(a) increase  (b) decrease  (c) stay the same

Angular momentum

The angular momentum is due to the girl, the merry-go-round and the ball. 

\[ L_{\text{NET}} = L_{\text{MGR}} + L_{\text{GIRL}} + L_{\text{BALL}} \]

Initial: 

\[ L_{\text{BALL}} = I \omega = \left( mR^2 \right) \left( \frac{v}{R} \right) = mvR \]

Final: 

\[ L_{\text{BALL}} = mvR \quad \text{same} \]
L total must be conserved:
Changing orientation of L_wheel will result in change of L_student_stool

Chopper needs the back wheel both for angular momentum balance as well as for torque compensation due to air “resistance”
Summary of Dynamics

- Dynamics (cause – effect)
  - Force – Linear acceleration
    - Change in linear momentum, $F_{net} = \frac{dp}{dt}$
  - Torque – angular acceleration
    - Change in angular momentum, $td\vec{L}/dt$

- Conservation of momentum
  - When net external force = zero

- Conservation of angular momentum
  - When net external torque = zero

- Conservation of Energy
  - Mechanical energy - kinetic + potential
  - Non-conservative forces e.g. friction
    - Energy lost to the environment, e.g., heat