Lecture 19

Goals:

- Chapter 14
  - Interrelate the physics and mathematics of oscillations.
  - Draw and interpret oscillatory graphs.
  - Learn the concepts of phase and phase constant.
  - Understand and use energy conservation in oscillatory systems.
  - Understand the basic ideas of damping and resonance.

Phase Contrast Microscope

Epithelial cell in brightfield (BF) using a 40x lens (NA 0.75) (left) and with phase contrast using a DL Plan Achromat 40x (NA 0.65) (right).

A green interference filter is used for both images.

Periodic Motion is everywhere

Examples of periodic motion

- Earth around the sun
- Elastic ball bouncing up and down
- Quartz crystal in your watch, computer clock, iPod clock, etc.
Periodic Motion is everywhere

Examples of periodic motion

- Heart beat
  In taking your pulse, you count 70.0 heartbeats in 1 min.

  What is the period, in seconds, of your heart's oscillations?
  Period is the time for one oscillation
  \[ T = \frac{60 \text{ sec}}{70.0} = 0.86 \text{ s} \]

- What is the frequency?
  \[ f = \frac{1}{T} = 1.17 \text{ Hz} \]

A special kind of periodic oscillator: Harmonic oscillator

What do all “harmonic oscillators” have in common?

1. A position of equilibrium
2. A restoring force, which must be linear
   [Hooke’s law spring \( F = -k x \)
   (In a pendulum the behavior only linear for small angles: \( \sin \theta \) where \( \theta = s / L \)] In this limit we have: \( F = -ks \) with \( k = mg/L \)
3. Inertia
4. The drag forces are reasonably small
Simple Harmonic Motion (SHM)

• In Simple Harmonic Motion the restoring force on the mass is linear, that is, exactly proportional to the displacement of the mass from rest position

• Hooke’s Law: \( F = -kx \)

If \( k \gg m \) \( \rightarrow \) rapid oscillations \( \leftrightarrow \) large frequency

If \( k \ll m \) \( \rightarrow \) slow oscillations \( \leftrightarrow \) low frequency

Simple Harmonic Motion (SHM)

• We know that if we stretch a spring with a mass on the end and let it go the mass will, if there is no friction, ….do something

1. Pull block to the right until \( x = A \)

2. After the block is released from \( x = A \), it will

A: remain at rest
B: move to the left until it reaches equilibrium and stop there
C: move to the left until it reaches \( x = -A \) and stop there
D: move to the left until it reaches \( x = -A \) and then begin to move to the right
Simple Harmonic Motion (SHM)

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This oscillation is called Simple Harmonic Motion

Simple Harmonic Motion (SHM)

- The time it takes the block to complete one cycle is called the period. Usually, the period is denoted \( T \) and is measured in seconds.

- The frequency, denoted \( f \), is the number of cycles that are completed per unit of time: \( f = 1 / T \).
  In SI units, \( f \) is measured in inverse seconds, or hertz (Hz).

- If the period is doubled, the frequency is
  
  A. unchanged
  B. doubled
  C. halved
Simple Harmonic Motion (SHM)

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Simple Harmonic Motion (SHM)

• An oscillating object takes 0.10 s to complete one cycle; that is, its period is 0.10 s.
• What is its frequency $f$?

Express your answer in hertz.

$f = 1 / T = 10$ Hz
Simple Harmonic Motion

- Note in the (x,t) graph that the vertical axis represents the x coordinate of the oscillating object, and the horizontal axis represents time.

Which points on the x axis are located a displacement A from the equilibrium position?

A. R only
B. Q only
C. both R and Q

Simple Harmonic Motion

- Suppose that the period is T.
- Which of the following points on the t axis are separated by the time interval T?
  A. K and L
  B. K and M
  C. K and P
  D. L and N
  E. M and P
Simple Harmonic Motion

- Now assume that the t coordinate of point K is 0.0050 s.
- What is the period T, in seconds?

- How much time t does the block take to travel from the point of maximum displacement to the opposite point of maximum displacement?

\[ T = 0.020 \text{ s} \]

\[ t = 0.010 \text{ s} \]
Simple Harmonic Motion

- Now assume that the x coordinate of point R is 0.12 m.
- What total distance \( d \) does the object cover during one period of oscillation?
  \[ d = 0.48 \text{ m} \]
- What distance \( d \) does the object cover between the moments labeled K and N on the graph?
  \[ d = 0.36 \text{ m} \]

SHM Dynamics: Newton’s Laws still apply

- At any given instant we know that \( F = ma \) must be true.
- But in this case \( F = -kx \) and \( ma = m\frac{d^2x}{dt^2} \)
- So: \(-kx = ma = m\frac{d^2x}{dt^2}\)

\[ \frac{d^2 x}{dt^2} = -\frac{k}{m} x \]

a differential equation for \( x(t) \)!

“Simple approach”, guess a solution and see if it works!
SHM Solution...

- Try either \( \cos(\omega t) \) or \( \sin(\omega t) \)
- Below is a drawing of \( A \cos(\omega t) \)
- where \( A = \) amplitude of oscillation

\[
\begin{align*}
T &= \frac{2\pi}{\omega} \\
A &= \text{amplitude of oscillation}
\end{align*}
\]

- [with \( \omega = (k/m)^{\frac{1}{2}} \) and \( \omega = 2\pi f = 2\pi / T \)]
- Both sin and cosine work so need to include both

Combining sin and cosine solutions

\[
x(t) = B \cos \omega t + C \sin \omega t \\
= A \cos (\omega t + \phi) \\
= A (\cos \omega t \cos \phi - \sin \omega t \sin \phi) \\
=A \cos \phi \cos \omega t - A \sin \phi \sin \omega t)
\]

Notice that \( B = A \cos \phi \)  \( C = -A \sin \phi \)  \( \tan \phi = -C/B \)

Use “initial conditions” to determine phase \( \phi \)!
Energy of the Spring-Mass System

We know enough to discuss the mechanical energy of the oscillating mass on a spring.

\[ x(t) = A \cos(\omega t + \phi) \]

If \( x(t) \) is displacement from equilibrium, then potential energy is

\[ U(t) = \frac{1}{2} k x(t)^2 = \frac{1}{2} k A^2 \cos^2(\omega t + \phi) \]

\[ v(t) = \frac{dx}{dt} \rightarrow v(t) = A \omega \left( -\sin(\omega t + \phi) \right) \]

And so the kinetic energy is just \( \frac{1}{2} m v(t)^2 \)

\[ K(t) = \frac{1}{2} m v(t)^2 = \frac{1}{2} m (A \omega)^2 \sin^2(\omega t + \phi) \]

Finally,

\[ a(t) = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi) \]

Energy of the Spring-Mass System

\[
\begin{align*}
x(t) &= A \cos(\omega t + \phi) \\
v(t) &= -\omega A \sin(\omega t + \phi) \\
a(t) &= -\omega^2 A \cos(\omega t + \phi)
\end{align*}
\]

Kinetic energy is always

\[ K = \frac{1}{2} m v^2 = \frac{1}{2} m (\omega A)^2 \sin^2(\omega t + \phi) \]

Potential energy of a spring is,

\[ U = \frac{1}{2} k x^2 = \frac{1}{2} k A^2 \cos^2(\omega t + \phi) \]

And \( \omega^2 = k / m \) or \( k = m \omega^2 \)

\[ U = \frac{1}{2} m \omega^2 A^2 \cos^2(\omega t + \phi) \]
Energy of the Spring-Mass System

\[ x(t) = A \cos(\omega t + \phi) \]
\[ v(t) = -\omega A \sin(\omega t + \phi) \]
\[ a(t) = -\omega^2 A \cos(\omega t + \phi) \]

And the mechanical energy is

\[ K + U = \frac{1}{2} m \omega^2 A^2 \cos^2(\omega t + \phi) + \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \phi) \]

\[ K + U = \frac{1}{2} m \omega^2 A^2 \left[ \cos^2(\omega t + \phi) + \sin^2(\omega t + \phi) \right] \]

\[ K + U = \frac{1}{2} m \omega^2 A^2 = \frac{1}{2} k A^2 \quad \text{which is constant} \]

Energy of the Spring-Mass System

So \( E = K + U = \text{constant} = \frac{1}{2} k A^2 \)

\[ \omega = \sqrt{\frac{k}{m}} \Rightarrow \omega^2 = \frac{k}{m} \]

At maximum displacement \( K = 0 \) and \( U = \frac{1}{2} k A^2 \)
and acceleration has its maximum (or minimum)

At the equilibrium position \( K = \frac{1}{2} k A^2 = \frac{1}{2} m v^2 \) and \( U = 0 \)
SHM So Far

- The most general solution is \( x = A \cos(\omega t + \phi) \)
  where \( A = \) amplitude
  \( \omega = \) (angular) frequency
  \( \phi = \) phase constant
- For SHM without friction, \( \omega = \sqrt{\frac{k}{m}} \)
  - The frequency does not depend on the amplitude!
  - This is true of all simple harmonic motion!
- The oscillation occurs around the equilibrium point where the force is zero!
- Energy is a constant, it transfers between potential and kinetic

The Simple Pendulum

- A pendulum is made by suspending a mass \( m \) at the end of a string of length \( L \). Find the frequency of oscillation for small displacements.
  \[ \Sigma F_y = ma_c = T - mg \cos(\theta) = m \frac{v^2}{L} \]
  \[ \Sigma F_x = ma_x = -mg \sin(\theta) \]
  If \( \theta \) small then \( x \equiv L \theta \) and \( \sin(\theta) \equiv \theta \)
  \[ \frac{dx}{dt} = L \frac{d\theta}{dt} \]
  \[ a_x = \frac{d^2x}{dt^2} = L \frac{d^2\theta}{dt^2} \]
  so \( a_x = -g \theta = L \frac{d^2\theta}{dt^2} \rightarrow L \frac{d^2\theta}{dt^2} - g \theta = 0 \)
  and \( \theta = \theta_0 \cos(\omega t + \phi) \) or \( \theta = \theta_0 \sin(\omega t + \phi) \)
  with \( \omega = \sqrt{\frac{g}{L}} \).
Velocity and Acceleration

Position: \( x(t) = A \cos(\omega t + \phi) \)
Velocity: \( v(t) = -\omega A \sin(\omega t + \phi) \)
Acceleration: \( a(t) = -\omega^2 A \cos(\omega t + \phi) \)

by taking derivatives, since:
\[
\begin{align*}
\frac{dv(t)}{dt} &= \frac{dx(t)}{dt} \\
\frac{da(t)}{dt} &= \frac{dv(t)}{dt}
\end{align*}
\]

\[
\begin{align*}
x_{\text{max}} &= A \\
v_{\text{max}} &= \omega A \\
a_{\text{max}} &= \omega^2 A
\end{align*}
\]

Lecture 19

• Assignment
  ❖ HW8, Due Wednesday, Apr. 8th
  ❖ Thursday: Read through Chapter 15.4