Mid Term Exam 3  
Phys 248  
Apr 24, 2009

Print your name and ID number clearly above.  
To receive full credit you must show all your work. If you do not show your work you will only receive partial credit, even for a correct final answer.

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**Useful constants:**

- Planck constant $h = 6.626 \times 10^{-34} \text{ J s}$
- Reduced Planck constant $\hbar = 1.054 \times 10^{-34} \text{ J s}$
- Permittivity of free space: $\varepsilon_0 = 8.854 \times 10^{-12} \text{ C}^2 \text{N}^{-1} \text{m}^{-2}$
- $hc = 1240 \text{ eV nm}$, $\hbar c = 197.3 \text{ eV nm}$, $c = 3 \times 10^8 \text{ m/s}$
- $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$, $k \text{ e}^2 = 1.44 \text{ eV nm}$
- Permeability of free space: $\mu_0 = 4\pi \times 10^{-7} \text{ T} \cdot \text{A}^{-1} \text{m}$
- $m_e = \text{electron mass} = 9.11 \times 10^{-31} \text{ kg} = 0.511 \text{ MeV/c}^2$
- $e = \text{electron charge} = -1.6 \times 10^{-19} \text{ C}$
- $u = \text{atomic mass unit} = 1.660539 \times 10^{-27} \text{ kg} = 931.494 \text{ MeV/c}^2$
- $mp = \text{proton mass} = 938.27 \text{ MeV/c}^2 = 1.007276u$
- $mn = \text{neutron mass} = 939.57 \text{ MeV/c}^2 = 1.008665u$
Problem 1: Magnetic Forces and Fields.

(a) (10 points) A singly charged ion of mass \( m \) is accelerated from rest by a potential difference \( \Delta V \) and deflected by a uniform magnetic field \( B \) (perpendicular to the ion’s velocity) into a circular path of radius \( R \). A doubly charged ion of mass \( m' \) is accelerated through the same \( \Delta V \) and is deflected by the same magnetic field into a circular path of radius \( R' = 2R \). What is the ratio of the masses of the ions?

Solution:

For the singly charged ion, from \( q\Delta V = \frac{1}{2}mv^2 \), we have \( v = \sqrt{\frac{2q\Delta V}{m}} \).

The circular path is given by \( R = \frac{mv}{qB} = \sqrt{\frac{2\Delta V}{Bq}} \). For the doubly charged ion, we have \( v' = \sqrt{\frac{2q'\Delta V}{m'}} \) and \( R' = \frac{m'v'}{q'B} = \sqrt{\frac{2\Delta V}{Bq'}} \sqrt{\frac{m'}{q'}} \).

Therefore, \( \frac{R'}{R} = 2 = \sqrt{\frac{m'/q'}{m/q}} = \sqrt{\frac{m'}{2m}} \), and \( \frac{m'}{m} = 8 \).
(b) (15 points) An infinitely long wire with current $I=2\ A$ is curved as shown below. Suppose an electron is located at point P and is moving with velocity $\vec{v} = -v_0\hat{y}$ at time $t_0$, where $v_0 = 3 \times 10^4\ m/s$. Determine the magnetic force (magnitude and direction) on the electron at time $t=t_0$.

Solution:

The magnetic field at point P can be obtained using the Biot-Savart law: the result is $\vec{B} = \frac{\mu_0 I}{4\pi d}\hat{z} = 2 \times 10^{-6}\ T$. The magnetic force on the electron is $\vec{F} = q\vec{v} \times \vec{B} = |e|v_0\hat{y} \times \frac{\mu_0 I}{4\pi d}\hat{z} = \frac{|e|v_0\mu_0 I}{4\pi d}\hat{x} = 9.6 \times 10^{-21}\ N\hat{x}$.
(c) (15 points) A long cylindrical shell with inner radius a and outer radius b carries total current $I_1$ with a uniform current density, as shown below. A long wire with current $I_2$ ($|I_2|=2|I_1|$) in the opposite direction is located at the center of the cylindrical shell. Compute the magnetic field (magnitude and direction) everywhere, and plot it as a function of the radius $r$.

Solution:

The magnetic field is most easily obtained using Ampere’s Law:

$$\int \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}.$$  

For $r<a$: $I_{enc} = I_2$, and hence $\int \mathbf{B} \cdot d\mathbf{l} = B 2\pi r = \mu_0 I_2$, such that

$$\mathbf{B}(r) = \frac{\mu_0 I_2}{2\pi r} \hat{\phi}.$$  

For $a<r<b$, $I_{enc} = I_2 + J\pi(r^2-a^2)$, and so

$$\mathbf{B}(r) = \frac{\mu_0 I_1}{2\pi r} \left(2 - \frac{r^2-a^2}{b^2-a^2}\right) \hat{\phi}.$$  

For $r>b$, the total current enclosed is just $|I_1|$, so

$$\mathbf{B}(r) = \frac{\mu_0 |I_1|}{2\pi r} \hat{\phi}.$$  

See the plot on the next page (numerical values chosen: $a=1$, $b=2$, $\mu_0 I_1/(2\pi) = 1$).
Problem 2: RC circuit.

In the circuit below, the switch S has been closed for a long time and the circuit carries a constant current. Take $C_1 = 6.00 \mu F$, $C_2 = 2.00 \mu F$, $R_1 = 2.00 \, k\Omega$, and $R_2 = 5.00 \, k\Omega$. The power delivered to $R_2$ is 3.20 W.

(a) (10 points) Find the charge on $C_1$ and $C_2$.

(b) (10 points) Now the switch is opened. After equilibrium is reached in the circuit, what is the charge on $C_1$ and $C_2$?

Suppose now the battery is disconnected from the circuit while S is kept open, such that the two capacitors fully discharge.

(c) (10 points) Determine the total charge stored on the capacitors as a function of time.

(d) (10 points) Compute the total energy dissipated during the discharge process. [Hint in c-d: reduce the circuit to a circuit with an equivalent resistor in series together with an equivalent capacitor.]
Solutions:
(a) When the switch is closed since a long time the capacitors are fully charged and they behave like open circuits, so the current flows in the two resistors that are in series. The power delivered to $R_2$ is: $P = R_2I^2$, hence $I = \sqrt{\frac{P}{R_2}} = 25.30\,mA$. The potential difference on $R_1$ is: $\Delta V_1 = R_1I = 50.60\,V$. The charge on $C_1$ is $Q_1 = C_1\Delta V_1 = 303.60\mu C$ and on $C_2$ it is $Q_2 = C_2\Delta V_2 = C_2R_2I = 253.00\mu C$. The electromotive force or potential difference at the battery terminals is $\varepsilon = \Delta V_1 + \Delta V_2 = (R_1 + R_2)I = 177.10\,V$.

(b) After a long time the switch is opened the current drops to zero because the two capacitors are fully charged and they behave like open circuits so no current will flow in $R_1$ and $R_2$. The charge on the capacitors has changed and the potential difference across $R_1$ and $R_2$ is zero because the current is zero. Hence: $Q_1' = C_1\varepsilon = 1.06\,mC$ and $Q_2' = C_2\varepsilon = 0.35\,mC$ since the potential difference of the battery is appears across each capacitor (they are in parallel with the battery).

(c)-(d). Once the battery is disconnected, the two capacitors are still in parallel and still there is no current in the circuit. Hence, the charge on the capacitors stays constant and the sum has the value $Q_1 + Q_2$ computed in (b). Since there is no flow of charge there is no dissipation, so the dissipated energy is zero.

The hint to consider an equivalent circuit with two capacitors and two resistors in series is useful to calculate the time constant $\tau = R_{eq}C_{eq}$, where $R_{eq} = 7\,k\Omega$, and $C_{eq} = \left(\frac{C_1C_2}{C_1 + C_2}\right) = \frac{12}{8}\mu F = 1.5\mu F$. The value of the time constant is useful to demonstrate mathematically what is written above from the equation of the circuit, $\frac{Q_1}{C_1} - IR_1 - \frac{Q_2}{C_2} - IR_2 = 0$. 
Problem 3: Quantum and nuclear physics.

(a) (10 points) A hydrogen atom in the 3s state has the following normalized wave function:

$$\psi_{3s}(r) = \frac{1}{\sqrt{\pi}} \left( \frac{1}{3a_0} \right)^{3/2} \left( 1 - \frac{2}{3} \frac{r}{a_0} + \frac{2}{27} \left( \frac{r}{a_0} \right)^2 \right) e^{-r/(3a_0)}.$$

What is the probability of finding the electron in the range $$\Delta r = 0.01a_0$$ of $$r = 2a_0$$, where $$a_0$$ is the Bohr radius?

Solution:

The probability of finding the electron in this range is given by

$$\text{Prob} = P(r = 2a_0) \Delta r,$$

where the radial probability density for this state is $$P(r) = 4\pi r^2 |\psi_{3s}(r)|^2$$. Evaluating it, one obtains $$\text{Prob} = 2.14 \times 10^{-6}$$. 
(b) (10 points) You are given two nuclides, X and Y. Initially, there are 2.5 times as many nuclei of type X as there are of type Y. Three days later, there are 4.20 times as many type X nuclei as type Y nuclei. If the half-life of the type Y nuclei is 1.60 days, what is the half-life of the type X nuclei?

Solution:

The decay is governed by \( N_i(t) = N_{i0} e^{-\lambda_i t} \), where for each nuclide, \( \lambda_i = \frac{\ln 2}{T_{1/2_i}} \). Given that \( N_{X0} = 2.5 N_{Y0} \), \( N_X(t = 3d) = 4.2 N_Y(t = 3d) \), and

\[
\lambda_Y = \frac{\ln 2}{T_{1/2_Y}} = \frac{\ln 2}{1.6d},
\]
we obtain the condition \( e^{-\lambda_x 3d} = \frac{4.2}{2.5} e^{-3\ln 2/1.6} \). Taking the \( \ln \) of both sides, we have

\[
-\lambda_x (3d) = \ln \frac{4.2}{2.5} - \frac{3\ln 2}{1.6} = -0.780857 \equiv -\alpha,
\]

such that \( T_{1/2_X} = \frac{3\ln 2}{\alpha} = 2.66d \).

(c) (10 points) Write down the explicit reaction and determine the Q value (in MeV) associated with the spontaneous fission of \( ^{236}\text{U} \) (Z=92) into \( ^{90}\text{Rb} \) (Z=37) and \( ^{143}\text{Cs} \) (Z=55). The masses of \( ^{236}\text{U} \), \( ^{90}\text{Rb} \), and \( ^{143}\text{Cs} \) are 236.045568u, 89.914809u and 142.92733u.

Solution:
The reaction proceeds as

\[
^{236}\text{U} \rightarrow ^{90}\text{Rb} + ^{143}\text{Cs} + 3n.
\]
The Q value is

\[
Q = (m_U - m_{Rb} - m_{Cs} - 3m_n)c^2.\]
Evaluating it yields \( Q = 165.56 \text{ MeV} \).