HW3 Solutions
Notice numbers may change randomly in your assignments and you may have to recalculate solutions for your specific case.

1. T&L 5.P.01
(a) What is the de Broglie wavelength of a 1 g mass moving at a speed of 1 m per year?
(b) What should be the speed of such a mass if its de Broglie wavelength is to be 1 cm?

Solution:

\[
v = \frac{1\text{ m/yr}}{3.17 \times 10^{-8} \text{ m/s}}
\]

(a)\[
p = \frac{h}{\lambda} = mv \Rightarrow \lambda = \frac{6.63 \times 10^{-34}}{10^{-3} \times 3.17 \times 10^{-8}} = \frac{h}{mv} = 2.09 \times 10^{-23} \text{ m}
\]

(b)\[
v = \frac{h}{m\lambda} = \frac{6.63 \times 10^{-34}}{10^{-3} \times 10^{-2}} = 6.63 \times 10^{-29} \text{ m/s}
\]

2. T&L 5.P.09
Compute the wavelength of a cosmic ray proton whose kinetic energy is each of the following.
(a) 2 GeV
(b) 200 GeV

Solution:

At these energies the electron is relativistic:
\[
E^2 = (pc)^2 + (mc^2)^2 = (pc)^2 + E_0^2 = (E_k + E_0)^2
\]
\[
p = \frac{(2E_kE_0 + E_k^2)^{1/2}}{c}
\]
\[
\lambda = \frac{hc}{(2E_kE_0 + E_k^2)^{1/2}}
\]
\[
E_0 = mc^2 = 0.938 \text{ MeV}
\]

(a)\[
\lambda = \frac{1240}{\sqrt{2 \times 0.938 \times 10^9 \times 2 \times 10^9 + (2 \times 10^9)^2}} = 0.445 \text{ fm}
\]

(b)\[
\lambda = \frac{1240}{\sqrt{2 \times 0.938 \times 10^9 \times 200 \times 10^9 + (200 \times 10^9)^2}} = 6.17 \times 10^{-3} \text{ fm}
\]
3. T&L 5.P.13
A certain crystal has a set of planes 0.30 nm apart. A beam of neutrons strikes the crystal at normal incidence and the first maximum of the diffraction pattern occurs at $\phi = 42^\circ$. What is the de Broglie wavelength of the neutrons? What is the kinetic energy?

**Solution:**
Let’s consider the geometry of this Davisson and Germer experiment with the normal incident beam on the atom planes of the crystal: the path difference between two beams shown in the figure is

$$d = t \cos(42\text{deg})$$

$$\Delta r = t + d$$

And the constructive interference condition on the phase difference is:

$$\Delta \phi = \frac{2\pi}{\lambda} \Delta r = \frac{2\pi}{\lambda} (1 + \cos 42^\circ) = 2m\pi, m = 1, 2, ...$$

The first maximum is for $m = 0$ =>

$$\frac{2\pi}{\lambda} (1 + \cos 42^\circ) = 2\pi \Rightarrow \lambda = t (1 + \cos 42^\circ) = 0.523 \text{nm}$$

Neutrons are not relativistic in the Davisson and Germer experiment:

$$E_k = \frac{1}{2} mv^2 = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} \Rightarrow \lambda = \frac{hc}{\sqrt{2mc^2E_k}} \Rightarrow E_k = \frac{(hc)^2}{2mc^2\lambda^2} = \frac{1240^2}{2 \times 0.938 \times 10^9 \times 0.523^2} = 3.0 \times 10^{-3} \text{eV}$$
4. T&L 5.P.17
Two harmonic waves travel simultaneously along a long wire. Their wave functions are $y_1 = 0.002\cos (8.0x - 400t)$ and $y_2 = 0.002\cos (7.6x - 380t)$, where $y$ and $x$ are in meters and $t$ in seconds.
(a) Write the wave function for the resultant wave in form of Equation 5-15.

$$y(x, t) = A\cos (Bx - Ct) \cos (Dx - Et)$$

(b) What is the phase velocity of the resultant wave?
(c) What is the group velocity?
(d) Calculate the range $\Delta x$ between successive zeros of the group. Relate $\Delta x$ to $\Delta k$.

**Solution:**
(a) 
$$y(x, t) = B\cos(k_1x - \omega_1t) + B\cos(k_2x - \omega_2t) =
= 2B\cos\left(\frac{k_1x - \omega_1t + k_2x - \omega_2t}{2}\right)\cos\left(\frac{k_1x - \omega_1t - k_2x + \omega_2t}{2}\right) =
= 2B\cos\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{k_1 - k_2}{2}x - \frac{\omega_1 - \omega_2}{2}t\right) =
= 2B\cos\left(\frac{\Delta k}{2}x - \frac{\Delta \omega}{2}t\right)\cos(\bar{k}x - \bar{\omega}t)$$

where 
$$\bar{k} = \frac{k_1 + k_2}{2}$$
$$\bar{\omega} = \frac{\omega_1 + \omega_2}{2}$$

and we used the trigonometric identity:
$$\cos a + \cos b = 2\cos\left(\frac{a + b}{2}\right)\cos\left(\frac{a - b}{2}\right)$$

Hence:
\[ A = 0.004 \text{m} \]
\[ B = \frac{k_1 - k_2}{2} = \frac{\Delta k}{2} = \frac{8.0 - 7.6}{2} = \frac{0.4}{2} = 0.2 \text{m}^{-1} \]
\[ C = \frac{\omega_1 - \omega_2}{2} = \frac{\Delta \omega}{2} = \frac{400 - 380}{2} = 10 \text{ rad/s} \]
\[ D = \frac{k_1 + k_2}{2} = \bar{k} = \frac{8.0 + 7.6}{2} = 7.8 \text{m}^{-1} \]
\[ E = \frac{\omega_1 + \omega_2}{2} = \bar{\omega} = \frac{400 + 380}{2} = 390 \text{ rad/s} \]

\[ y(x, t) = 0.004m \cos \left( \frac{0.2x}{m} - 10t/s \right) \cos \left( \frac{7.8x}{m} - 390t/s \right) \]

(b) Phase velocity: \[ v_p = \frac{\bar{\omega}}{\bar{k}} = \frac{E}{D} = 50 \text{m/s} \]

(c) Group velocity: \[ v_g = \frac{\Delta \omega}{\Delta k} = \frac{C}{B} = 50 \text{m/s} \]

Since the phase and group velocity are the same the wave is propagating in a non-dispersive medium.

(d) For the same instant of time \( t \), there will be a zero in \( 0.2x = \frac{\pi}{2} \) and \( 0.2x = \frac{3\pi}{2} \), hence successive zeroes in the envelope require

\[ 0.2 \Delta x = \frac{3\pi}{2} - \frac{\pi}{2} = \pi \Rightarrow \Delta x = \frac{\pi}{0.2} = 5\pi \text{m} \]

The uncertainty principle relation, since \( \Delta k = 0.4 \text{m}^{-1} \), is \( \Delta x \times \Delta k = 0.4 \times 5\pi = 2\pi \).

5. T&L 5.P.20

A certain standard tuning fork vibrates at 880 Hz. If the tuning fork is tapped, causing it to vibrate, then stopped 0.25 of a second later, what is the approximate range of frequencies contained in the sound pulse that reached your ear?

Solution:

The uncertainty principle says \( \Delta \omega \times \Delta t \approx 1 \Rightarrow \Delta f = \frac{\Delta \omega}{2\pi} = \frac{1}{2\pi \Delta t} = 0.6 \text{Hz} \)

6. T&L 5.P.25

The wave function describing a state of an electron confined to move along the \( x \) axis is given at time zero by the following equation:

\[ \Psi(x,0) = Ae^{-x^2/4\sigma^2} \]

Find the probability of finding the electron in a region \( dx \) centered
at each of the following locations.
(a) \( x = 0 \)
(b) \( x = \sigma \)
(c) \( x = 1.8\sigma \)

**Solution:**

The probability to find the electron in a region \( dx \) around \( x \) is:

\[
P(x)dx = |\Psi(x,0)|^2 dx = A^2 e^{-x^2/\sigma^2} dx
\]

(a) For \( x = 0 \):

\[
P(x)dx = |\Psi(0,0)|^2 dx = A^2 dx
\]

(b) \( x = \sigma \):

\[
P(x)dx = |\Psi(0,0)|^2 dx = A^2 e^{-1/2} dx = 0.607 A^2 dx
\]

(c) \( x = 1.8 \sigma \):

\[
P(x)dx = |\Psi(0,0)|^2 dx = A^2 e^{-1.8^2/2} dx = 0.198 A^2 dx
\]

(d)

7. T&L 5.P.38

A neutron in an atomic nucleus is bound to other neutrons and protons in the nucleus by the strong force when it comes within about 1 fm of another particle. What is the approximate kinetic energy of a neutron that is localized to within such a region?

What would be the corresponding energy of an electron localized to within such a region?

**Solution:**

\[
\Delta p \Delta x \approx \hbar \rightarrow \Delta p \approx \hbar/\Delta x = \hbar/1 \text{fm}
\]

Thus \((\Delta p)^2 = \frac{\hbar^2}{(1 \text{fm})^2}\)

The neutron is not relativistic so:

\[
\bar{E} = \frac{p^2}{2m} \geq \frac{\hbar^2}{2m(\Delta x)^2} = \frac{(hc)^2}{2mc^2(\Delta x)^2} = \frac{(197.3 eV \cdot nm)^2}{2 \times 939 \times 10^6 eV \times (10^{-6} \text{nm})^2} = 20.7 MeV
\]

The electron is relativistic because the classical kinetic energy \(p^2/2m\) is much larger than the rest energy, so:
\[ E^2 = (pc)^2 + (m_e c^2)^2 \]
\[ E^2 = \left(\frac{\hbar c}{fm}\right)^2 + (m_e c^2)^2 \]
\[ E^2 = (197.30 \text{ MeV} \cdot fm)^2 / (fm)^2 + (0.511 \text{ MeV})^2 \]
\[ E^2 = 3.8928 \times 10^4 \text{ MeV}^2 \]
\[ \therefore E = 197.30 \text{ MeV} \]

and \[ E_\kappa = 197.30 \text{ MeV} - 0.511 \text{ MeV} = 196.79 \text{ MeV} \]

8. T&L 5.P.45
A proton and a 10 g bullet each move with a speed of 510 m/s, measured with an uncertainty of 0.04 percent. If the measurements of their respective positions are made simultaneous with the speed measurements, what is the minimum uncertainty possible in the position measurements?

**Solution:**
Given the uncertainty principle:
\[ \Delta x \Delta p \geq \frac{\hbar}{2} \Rightarrow \Delta x \geq \frac{\hbar}{2m \Delta v} = \frac{1.054 \times 10^{-34}}{2 \times 1.673 \times 10^{-32} \times 0.04/100 \times 510} = 1.55 \times 10^{-7} \text{ m (proton)} \]
\[ \Delta x \Delta p \geq \frac{\hbar}{2} \Rightarrow \Delta x \geq \frac{\hbar}{2m \Delta v} = \frac{1.054 \times 10^{-34}}{2 \times 10 \times 10^{-3} \times 0.04/100 \times 510} = 2.59 \times 10^{-32} \text{ m (bullet)} \]

A particle is in an infinite square well of size \( L \). Calculate the ground state energy for the following conditions.
(a) The particle is a proton and \( L = 0.1 \) nm, a possible size of a molecule.
(b) The particle is a proton and \( L = 1 \) fm, a possible size of a nucleus.
Solution:

(a) The ground state of an infinite well is \( E_1 = \frac{\hbar^2}{8mL^2} = \frac{(\hbar c)^2}{8mc^2L^2} \)

For \( m = m_p, L = 0.1 \, \text{nm} \):

\[
E_1 = \frac{(1240 \, \text{MeV} \cdot \text{fm})^2}{8(938.3 \times 10^6 \, \text{eV}) (0.1 \, \text{nm})^2} = 0.021 \, \text{eV}
\]

(b) For \( m = m_p, L = 1 \, \text{fm} \):

\[
E_1 = \frac{(1240 \, \text{MeV} \cdot \text{fm})^2}{8(938.3 \times 10^6 \, \text{eV}) (1 \, \text{fm})^2} = 205 \, \text{MeV}
\]


A mass of \( 10^{-6} \, \text{g} \) is moving with a speed of about \( 10^{-1} \, \text{cm/s} \) in a box of length 1 cm. Treating this as a one-dimensional infinite square well, calculate the approximate value of the quantum number \( n \).

Solution:

Using \( E = \frac{1}{2}mv^2 = \frac{n^2\pi^2\hbar^2}{2mL^2} \), we obtain \( n^2 = \frac{m^2v^2L^2}{\pi^2\hbar^2} \), and thus

\[
n = \frac{mvL}{\pi\hbar} = \frac{(10^{-9} \, \text{kg})(10^{-3} \, \text{m/s})(10^{-2} \, \text{m})}{\pi(1.055 \times 10^{-34} \, \text{Js})} = 3.02 \times 10^{19}.
\]

11. (similar to T&L Example 6-3)

A particle is in the first excited state of an infinite square well of size \( L \), with \( V(x)=0 \) between \( x=0 \) and \( x=L \), and \( V(x)=\infty \) elsewhere. Compute the probability of finding the particle in the following intervals:

(a) The interval between \( x=0 \) and \( x=L/2 \).

(b) The interval between \( x=3L/4 \) and \( x=L \).

Solution:

The wavefunction of the particle in the first excited state is \( \psi(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{2\pi x}{L} \right) \), such that the probability of finding the particle between two points \( a \) and \( b \) is

\[
\int_a^b \psi(x)^2 \, dx.
\]
Therefore, for part (a), we have \( \frac{2}{L} \int_0^{\frac{L}{2}} \sin^2 \left( \frac{2\pi x}{L} \right) dx = \frac{1}{2} \), and for part (b),
\[
\frac{2}{L} \int_{\frac{3L}{4}}^{L} \sin^2 \left( \frac{2\pi x}{L} \right) dx = \frac{1}{4}.
\]
Note that this problem differs from Example 6-3 in here we are considering the first excited state, whereas Example 6-3 considers the ground state. The probability of finding the particle between \( 3L/4 \) and \( L \) is much larger for the first excited state than for the ground state, because of the very different shape of the wavefunctions.

12. T&L 6.P.16
The wavelength of light emitted by a ruby laser is 694.3 nm. Assuming that the emission of a photon of this wavelength accompanies the transition of an electron from the \( n=2 \) to the \( n=1 \) level of an infinite square well (with \( V(x)=0 \) between \( x=0 \) and \( x=L \), and \( V(x)=\text{infinity otherwise} \)), compute the length \( L \) of the well (in nm).

**Solution:**
The energy levels of the infinite square well are \( E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} = \frac{n^2 \hbar^2}{8mL^2} \). In the transition between the energy levels, the wavelength of the emitted photon satisfies
\[
\Delta E = E_2 - E_1 = \frac{\hbar c}{\lambda}.
\]
Therefore, we have \( E_2 - E_1 = 4E_1 - E_1 = 3E_1 = 3 \frac{\hbar^2}{8mL^2} = \frac{\hbar c}{\lambda} \), and
\[
L = \sqrt{\frac{3\hbar c\lambda}{8mc^2}} = \sqrt{\frac{3(1240)(694.3)}{8(.511 \times 10^6)}} nm = 0.795 nm.
\]